Mon. Not. R. Astron. Soc. **000**, 1–7 (2005) Printed 4 August 2005 (MN LaTeX style file v2.2)

Compressible magnetohydrodynamic turbulence in molecular clouds

Lucy Liuxuan Zhang,[1,3⋆] Christopher D. Matzner,[2†] Ue-Li Pen,[1‡]
[1] *Canadian Institute for Theoretical Astrophysics, University of Toronto, M5S 3H8, Canada*
[2] *Department of Astronomy and Astrophysics, University of Toronto, M5S 3H8, Canada*
[3] *National Astronomical Observatories, Chinese Academy of Sciences, Beijing 100012, China*

4 August 2005

ABSTRACT

We present numerical simulations of supersonic, magnetized, isothermal turbulence. To represent physical conditions within molecular clouds, our runs possess Mach numbers as high as 10 and have Alfvén Mach numbers around unity. We study the rate of energy dissipation as well as the turbulent spectrum, density distribution, and density-magnetic field correlation, as well as the properties of dense substructures whose collapse could produce stars. However, we omit gravity from our calculations.

Arguing that proper resolution requires the driving wavelength to exceed the grid spacing by at least the shock compression factor, we test this rule in convergence studies. We also vary the time coherence of the driving force, comparing rapidly fluctuating forcing (for comparison with prior work) with forcing that is coherent on its natural period. These comparisons, along with the high dynamic range of the current simulations (up to 1024^3 cells), permit us to separate physical properties of supersonic turbulence from numerical artifacts.

Key words: MHD – ISM: magnetic fields – turbulence – star formation

1 INTRODUCTION

Theoretical studies of star formation often rely on computer simulations to provide insight into the dynamics of the supersonically turbulent, magnetized molecular clouds from which stars form. Within these clouds, molecular line width corresponds to a one-dimensinoal velcoity dispersion that scales with map size as

$$\sigma \approx 0.72 \left(\frac{R}{\mathrm{pc}}\right)^{0.5} \ \mathrm{km\,s^{-1}} \qquad (1)$$

(Larson 1981; Solomon et al. 1987), and in many cases the Alfvén speed $v_A \equiv B/\sqrt{4\pi\rho}$ is inferred from Zeeman measurements to be comparable to σ (Crutcher 1999). In contrast, CO rotational cooling maintains temperatures in the range 10-30 K and thermal sound speeds $c_s \sim 0.2 - 0.3\ \mathrm{km\,s^{-1}}$ in the well-shielded regions to which star formation appears to be restricted (Onishi et al. 1998; Johnstone et al. 2004). Relevant Mach numbers $\mathcal{M} \equiv \sigma/c_s$ vary from about unity, for the dense "cores" that form individual stars ($R \simeq 0.07$ pc), to about 15 for the largest Milky Way clouds ($R \simeq$100 pc; Solomon et al. 1987). Alfvén Mach numbers ($\mathcal{M}_A \equiv \sigma/v_A$) are of order unity.

In addition to $\mathcal{M}$ and $\mathcal{M}_A$, the virial parameter

$$\alpha \equiv \frac{5\sigma^2 R}{GM}$$

(Bertoldi & McKee 1992) forms a third dimensionless physical parameter that measures the influence of gravity on cloud motions; it is about 1.1 for molecular clouds in the Solomon et al. (1987) survey, indicating that clouds are confined largely by self-gravity. It is also about unity for thermally supported prestellar cores ($\sigma \to c_s$ in this case) and for the massive clumps within molecular clouds that form star clusters. However Bertoldi & McKee (1992) find that many smaller compressions within molecular clouds are confined by pressure rather than gravity, as they have $\alpha \gg 1$.

Our work follows that of Stone et al. (1998, hereafter SOG) and Mac Low (1999, hereafter ML). Table 1 lists the dimensionless parameters, the coherence time t_c [1], the initial v_A/c_s, the saturated σ and the saturated $\mathcal{M}$, employed in their simulations. Here, v_A/c_s is related to another commonly used parameter $\beta \equiv P_{gas}/P_{mag} = \rho c_s^2/[B^2/2(4\pi)]$ by $(v_A/c_s)^2 = 2/\beta$. In Table 1, we define σ in terms of the total

⋆ Email: lxzhang@cita.utoronto.ca
† Email: matzner@cita.utoronto.ca
‡ Email: pen@cita.utoronto.ca

[1] The coherence time t_c will be defined in section 2.3.2.

mass M and the total kinetic energy E_k as

$$\sigma = \sqrt{\frac{2E_k}{3M}}. \quad (2)$$

Our choices for these parameters are discussed below. Like SOG and ML, we neglect gravity entirely ($\alpha \to \infty$). Although this prevents us from addressing any self-gravitating structure or the process of star formation itself, to resolve these processes would require special methods such as the addition of "sink particles" (e.g., Heitsch et al. 2001; Klessen et al. 2000). It is hoped that turbulence studied in isolation is still characteristic of a more realistic context.

Whereas physical properties of a turbulent fluid are characterized by its dimensionless parameters $\mathcal{M}$, $\mathcal{M}_A$ and α, several additional parameters are required to specify the *numerical* implementation of a given simulation:

- *Boundary conditions.* When studying turbulence, authors invariably employ either fixed and strictly periodic boundaries, or (for flows within disks) periodicity with an imposed shear. The existence of external flows, or of an overall contraction of the cloud, is incompatible with the former and quite restricted in the latter case. (A dilating background state could be assumed, but this has not been used for studying turbulence.) Furthermore, large-scale gradients and cloud boundaries cannot be incorporated if the internal dynamics are resolved with strictly periodic boundary conditions and a uniform initial state.

- *Forcing.* In the absence of shearing boundary conditions, turbulence must either be forced within the simulation volume or allowed to decay from an initial state. Although the focus is typically on the properties of turbulence rather than how it is driven, these may not be entirely separable: large-scale shear, expanding photoionized regions, and protostellar jets will drive qualitatively different motions. We follow SOG and ML by employing a driving acceleration field $\boldsymbol{a}_d$ which is Gaussian random and peaks at a wavelength λ_d that is a fraction of the box width L. This leaves unanswered what value of λ_d best represents a particular form of driving. In addition, one must choose the correlation properties of this driving field, both in space – should it be incompressive ($\boldsymbol{\nabla} \cdot \boldsymbol{a}_d = 0$)? correlated with the magnetic field? – and in time – how frequently does the driving field change direction?

The drivers of turbulence within molecular clouds, be they protostellar winds or H II regions, derive either from newborn stars or from outside the cloud boundaries. Although the driving acceleration used in our simulations are more consistent with the former hypothesis, it is not correlated with regions of compression like star formation would be. In fact, we impose no correlation at all between the form of the driving field and the fluid it drives. Some correlation will naturally develop as the fluid responds to the forcing field.

Forcing introduces several dimensionless parameters:

-*Sample variance and box effects.* To be consistent with periodic boundary conditions, an integer number of forcing wavelengths must fit within the length L of the computational box. If the peak of the forcing spectrum has a wavelength λ_d, then

$$\frac{L}{\lambda_d}$$

determines both the number of independent wavelengths (and hence the "sample variance", in analogy with cosmological simulations), and the effect of periodicity on the motions at scale λ_d. To ensure a valid ensemble average over many driving cells, and to prevent large-scale periodicity from imprinting itself on the flow structures, one desires $L/\lambda_d \gg 1$. SOG and ML have used $L/\lambda_d = 4$ or 8 (with one exception; see Table 1).

-*Resolution.* Whereas turbulence in an incompressible fluid involves a cascade of energy to successively smaller scales, energy dissipation in highly compressive turbulence occurs at shocks. Conservative codes, like that used here, match shock jump conditions and shock geometries quite well. This fact lends credence to integral quantities, like the dissipation rate, derived from these codes – *provided* all physically relevant scales are at least a few times the resolution scale Δx. The driving scale is resolved if $\lambda_d \gg \Delta x$; but more importantly, a reliable simulation should resolve a slab, originally $\sim \lambda_d$ in width, that has been compressed by a shock of speed $\sim \sigma$. Isothermal shocks compress gas by $\mathcal{M}^2$ or $\sqrt{2}\mathcal{M}_{A\perp}$ (where $\mathcal{M}_{A\perp}$ includes only field components perpendicular to the shock normal) depending on whether thermal or magnetic support is important; the corresponding "resolution parameters" are

$$\Re \equiv \frac{\lambda_d}{\Delta x}\frac{c_s^2}{\sigma^2} \quad \text{and} \quad \Re_A \equiv \frac{\lambda_d}{\Delta x}\frac{v_A}{\sigma}.$$

A convergence study is required to demonstrate what values of $\Re$ and $\Re_A$ suffice for a given problem. Resolving shocks along the field or in unmagnetized gas requires $\Re \gg 1$, which is far more stringent than $\Re_A \gg 1$ for molecular clouds. These constraints burden the largest computers in existence, as one must have $L \gg 400\Delta x$ to satisfy $L/\lambda_d \geqslant 4$, and $\lambda_d \gg \mathcal{M}^2\Delta x$ when $\mathcal{M} \sim 10$. The slow convergence noted by SOG is presumably a consequence of this fact. Note that in Table 1, only the low-Mach number runs are adequately resolved by both criteria.

-*Coherence time.* The events that drive turbulence on a scale λ_d are likely to be explosive motions, such as protostellar outflows (Norman & Silk 1980; McKee 1989) or H II regions (Arons & Max 1975; Matzner 2002). Regardless of origin, these motions will render their energy to turbulence when they slow to $\sim \sigma$; they therefore persist for $\sim \lambda_d/\sigma$. In a numerical study one must choose the persistence time t_c of the driving field; the ratio

$$\frac{t_c \sigma}{\lambda_d}$$

is of order unity for physical sources. SOG change forcing fields every timestep ($t_c\sigma/\lambda_d \sim 10^{-2}$) whereas ML do not change it at all ($t_c\sigma/\lambda_d = \infty$). As presented in Table 1, their results are consistent within a factor of several despite this. We employ $t_c \simeq \lambda_d/\sigma$, an intermediate and physical choice.

- *Fluid physics.* For numerical convenience, most turbulence simulations to date treat molecular cloud gas as a single, perfectly conducting, strictly isothermal or at least barotropic fluid; so do we. (An exception is the work by Padoan et al. 2000.) In reality, ion-neutral drift produces an effective diffusivity, damping high frequency waves and allowing mass to cross field lines. Reconnection may occur naturally in turbulent flows (Lazarian & Vishniac 1999); it is a feature of numerical simulations also, due to numerical diffusivity. Although the thermal time is short, gas temper-

Table 1. Turbulence simulations by ML and SOG: Numerical and physical parameters

		Numerical parameters			Physical parameters		Resolution		Dissipation rate
Model	Reference	$\frac{L}{\lambda_d}$	$\frac{\lambda_d}{\Delta x}$	$\frac{t_c \sigma}{\lambda_d}$	$\frac{v_A}{c_s}$	$\mathcal{M}$	$\Re$	$\Re_A$	$\frac{\dot{E}\lambda_d}{M\sigma^3}$
HA8	ML	8	16	∞	0	1.0	15.3	—	2.9
HB8	ML	8	16	∞	0	1.4	7.8	—	3.2
HC2	ML	2	64	∞	0	4.5	3.2	—	1.4
HC4	ML	4	32	∞	0	2.8	4.0	—	2.7
HC8	ML	8	16	∞	0	2.2	3.4	—	3.0
HD8	ML	8	16	∞	0	3.2	1.6	—	2.9
HE2	ML	2	64	∞	0	8.6	0.86	—	2.0
HE4	ML	4	32	∞	0	6.7	0.71	—	2.0
HE8	ML	8	16	∞	0	4.7	0.73	—	3.1
MA4X	ML	4	32	∞	10	1.5	14.1	210	1.8
MA81	ML	8	16	∞	1	0.8	25.2	20	6.2
MC4X	ML	4	32	∞	10	2.6	4.9	120	3.7
MC45	ML	4	32	∞	5	2.5	5.1	64	4.0
MC41	ML	4	32	∞	1	2.6	4.7	12	3.5
MC85	ML	8	16	∞	5	1.7	5.5	47	6.3
MC81	ML	8	16	∞	1	1.9	4.5	8.5	4.7
A	SOG	8	32	0.024	10	2.9	3.7	110	4.9
B	SOG	8	32	0.022	3.2	2.8	4.1	6.1	5.7
C	SOG	8	32	0.023	1.0	2.9	3.7	11	5.0
D	SOG	8	32	0.026	0	3.2	3.1	—	3.8

atures are elevated in regions shielded by fewer than about four visual magnitudes (Tielens & Hollenbach 1985).

- *Algorithm.* The order of convergence of the numerical code determines how many grid cells must be devoted to reliably resolve a given physical scale, and to hold numerical diffusion to a given level. Sufficient values of $\Re$ and $\Re_A$ are therefore somewhat code-dependent. SOG and ML used the ZEUS code; we use an isothermal variant of the Total Variation Diminishing (TVD) code presented by Pen et al. (2003). Our code is faster than ZEUS and has been demonstrated to outperform ZEUS in tests of flux tube advection (Robert Fisher 2004, private communication). The unsplit Godunov code Athena (Gardiner & Stone 2005) has also been used to study molecular cloud turbulence, but the results of this study are unpublished.

2 NUMERICAL MODEL

2.1 Initial condition

The initial condition which we use is identical to that of SOG. We have a cubic, periodic box of fluid which starts at rest, with uniform density ρ_0, and threaded by a uniform magnetic field in the z-direction. The magnitude of the initial magnetic field is given by $b_0 \equiv B_0/\sqrt{4\pi} = \sqrt{2\rho_0 c_s^2/\beta}$. We use the isothermal approximation to model the gas dynamics in molecular clouds. In such a model, the sound speed c_s is constant in both space and time.

2.2 Advection scheme

We start with an MPI version of a 2$^{\rm nd}$ order, flux constraint transport, TVD, adiabatic MHD code (Pen et al. 2003) parallelized by Matthias Liebendörfer. We modify the code to study the dynamics of a magnetized fluid with an isothermal equation of state. The MHD equations for an isothermal fluid express the conservation of mass and momentum, as well as the magnetic flux freezing condition and the divergence free nature of the magnetic field. We use operator splitting between advection and driving. At each advection step, we solve the following system of partial differential equations in a cubical box with periodic boundary conditions:

$$\begin{aligned}
\partial_t \rho + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) &= 0; \\
\partial_t (\rho v_i) + \partial_j (\rho v_i v_j + P_* \delta_{ij} - b_i b_j) &= 0; \\
\partial_t \boldsymbol{b} &= \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{b}); \\
\boldsymbol{\nabla} \cdot \boldsymbol{b} &= 0; \\
P_* &= p + \frac{b^2}{2}.
\end{aligned}$$

Here, ρ is the density of the fluid, $\boldsymbol{v}$ the velocity, P_* the total pressure, $p = \rho c_s^2$ the pressure of the isothermal gas, and δ_{ij} the Kronecker delta symbol; $\boldsymbol{b}$ is shorthand for $\boldsymbol{B}/\sqrt{4\pi}$. The energy equation does not apply for a strictly isothermal fluid; but as discussed in the Appendix, it is nevertheless useful to introudce a compressional free energy that is dissipated at shocks.

In our TVD solver, we employ a smooth Minmod flux limiter (Hirsch 1990). As in all explicit Eulerian hydrodynamics schemes, the numerical time step must be below a threshold value to ensure numerical stability. Taking into account of turbulence driving, we adopt an advection time step of

$$\Delta t = \min\left[\frac{0.7\Delta x}{\max\left(|\boldsymbol{v}| + \sqrt{c_s^2 + 2b^2/\rho}\right)}, 0.5\ (t_d - t)\right]$$

where t_d is the time for the next driving, and t the current simulation time. The factor of 0.5 exists in the expression because our advection scheme makes use of dimensional splitting.

2.3 Turbulence driving schemes

We use two methods of driving in this work, types I and II, which are distinguished by the forcing spectrum, the coherence time t_c of the driving force and by the manner in which it is normalized. Despite these differences, the two schemes have several things in common.

In either case, we update the fluid velocity on regular intervals

$$\delta t = 10^{-3}\frac{L}{c_s} \tag{3}$$

by an amount

$$\delta \boldsymbol{v} = \boldsymbol{a}_d \cdot \delta t,$$

where $\boldsymbol{a}_d$ is the imposed acceleration field. Each $\delta \boldsymbol{v}$ is an independent realization of a Gaussian random field with some prescribed power spectrum. To avoid the use of parallel Fourier transform when constructing $\delta \boldsymbol{v}$, we generate a coarse (32^3) version of the global velocity perturbation field on each processor, and add the relevant region of $\delta \boldsymbol{v}$ to the local velocity field by interpolation. Remember that the density field ρ and the velocity field $\boldsymbol{v}$ live on the fine grid, whereas, $\delta \boldsymbol{v}$ and $\delta \boldsymbol{A}_{\rm new}$ are coarse-grid quantities. Whenever the two types of quantities appear in the same expression, we always interpolate the coarse-grid quantities onto the fine grid, and carry out operations there. In this subsection, we focus on the construction of $\delta \boldsymbol{v}$ carried out on each processor.

Both driving schemes require us to generate a divergenceless velocity field $\delta \boldsymbol{v}_{\rm new}$ with zero net momentum and a desired power spectrum $|\widetilde{\delta \boldsymbol{v}}(k)|^2$. We define k_{pk} as the value of wave number $k = 2\pi/\lambda$ at which $k^2|\widetilde{\delta \boldsymbol{v}}(k)|^2$ peaks. (Tildes indicate spatial Fourier transforms.) For this we construct a vector potential $\delta \boldsymbol{A}_{\rm new}$, each of whose components is obtained by convolving an independent Gaussian random white noise field with the power spectrum

$$|\widetilde{\delta \boldsymbol{A}}_{\rm new}(k)|^2 = \frac{|\widetilde{\delta \boldsymbol{v}}(k)|^2}{k^2}. \tag{4}$$

The resulting $\delta \boldsymbol{A}_{\rm new}$ is a Gaussian random field, with uncorrelated Cartesian components, whose power spectrum is steeper by two powers of k than that of the desired velocity perturbation field. The curl of $\delta \boldsymbol{A}_{\rm new}$ yields a divergenceless Gaussian random velocity field whose spectrum is $|\widetilde{\delta \boldsymbol{v}}(k)|^2$, which we adopt as $\delta \boldsymbol{v}_{\rm new}$.

In order to control the rate of energy input and of $\delta \boldsymbol{v}$ decoherence, we normalize or "mix"[2] $\delta \boldsymbol{v}_{\rm new}$ to obtain the velocity perturbation $\delta \boldsymbol{v}$. In the process of normalization and mixing, we also ensure that the generated velocity perturbation $\delta \boldsymbol{v}$ transfers no net momentum to the fluid; i.e., $\int \rho\, \delta \boldsymbol{v}\, d^3x = 0$.

2.3.1 *Type I driving*

Type I driving mimics the methodology of SOG. The spatial power spectrum of the driving velocity field is

$$|\widetilde{\delta \boldsymbol{v}}(k)|^2 \propto k^6 \exp\left(-8\frac{k}{k_{pk}}\right). \tag{5}$$

[2] "Mixing" is described in section 2.3.2.

Subtracting from $\delta \boldsymbol{v}_{\rm new}$ a uniform velocity $\delta \boldsymbol{v}_p$ chosen such that no net momentum is transferred, we then normalize the result to obtain an energy injection rate of precisely $10^3 \rho_0 L^2 c_s^3$ where ρ_0 is the average density. The normalization is achieved by multiplication by an overall constant $\phi(t)$, and requires the solution of a quadratic equation for the normalization factor; we choose a *random* root while noting that ML chose the larger root and that SOG did not specify their choice. The final velocity perturbation is

$$\delta \boldsymbol{v} = \phi\ (\delta \boldsymbol{v}_{new} - \delta \boldsymbol{v}_p)\,. \tag{6}$$

For type II driving, we define the coherence time as the perturbation time step; that is $t_c = \delta t$. Notice that ϕ varies at different perturbation time steps depending on $\delta \boldsymbol{v}_{new}$ and $\boldsymbol{v}$. Such energy normalization implies that the magnitude of the imposed acceleration field is dependent on the current state of the fluid.

2.3.2 *Type II driving*

Our second driving scheme differs in several ways from the first.

Its power spectrum,

$$|\widetilde{\delta \boldsymbol{v}}(k)|^2 \propto k^6 \exp\left(-4\frac{k^2}{k_{pk}^2}\right), \tag{7}$$

peaks more narrowly around k_{pk}. We introduced this alteration because the rate of energy dissipation is linearly proportional to k_{pk}, and we would like the forcing scale to be as well-defined as possible.

Although we generate a new velocity perturbation field $\delta \boldsymbol{v}_{\rm new}$ every $10^{-3}L/c_s$, as in type I driving, we do not simply adopt this as our velocity perturbation $\delta \boldsymbol{v}$. Instead, we require that $\delta \boldsymbol{v}$ possess a finite coherence time t_c which we control to be of order λ_d/σ. This is accomplished by mixing $\delta \boldsymbol{v}_{\rm new}$ with $\delta \boldsymbol{v}_{\rm old}$, the velocity perturbation from the previous driving step as follows:

$$\delta \boldsymbol{v} = \exp\left(-\frac{\delta t}{t_c}\right)\delta \boldsymbol{v}_{\rm old} + \phi\left[1 - \exp\left(-\frac{\delta t}{t_c}\right)\right]\delta \boldsymbol{v}_{\rm new} - \delta \boldsymbol{v}_p \tag{8}$$

where $\delta \boldsymbol{v}_p$ is again the uniform velocity field which must be subtracted to ensure incompressiveness of $\delta \boldsymbol{v}$. In type II driving, however, the normalization constant ϕ in equation (8) is constant throughout a simulation run; it controls the average input power $\langle \dot{E} \rangle_t$. Note that the spatial properties of $\delta \boldsymbol{v}_{\rm old}$ and $\delta \boldsymbol{v}_{\rm new}$ – their Gaussianity, spectrum, divergenceless and lack of net momentum – are inherited by $\delta \boldsymbol{v}$ in this procedure.

In this scheme, the magnitude of the acceleration field is constant in time but the energy input rate is only constant on average in the saturated state. We present the values of ϕ and the estimated energy injection rate $\dot{\mathcal{E}}$ for type II driving in Table 3. The saturated averages of energy injection rate (averaged over many crossing periods λ_d/σ after saturation) are listed in Table 2.

As $t_c \to 0$, not only is the velocity perturbation field at any given time Gaussian random in space, each component of $\delta \boldsymbol{v}$ at any given point in space also becomes Gaussian random in time.

3 SIMULATION RESULTS

The simulations are run on CITA's beowulf cluster, McKenzie, which has 528 2.4GHz Xeon processors. Each of the simulations presented in this paper takes about a few days to run.
[Lucy: insert 2D pictures]
[Lucy: insert 1D plots – energy, DVps, $\Delta E_k(\mathbf{t})$, t_c]

3.1 Turbulent energy: equipartition and dissipation rate

Our simulations are presented in Table 2 along with results for the degree of equipartition among components of the turbulent energy and the energy dissipation rate in each simulation. In each case, the results presented are representative of the saturated state of the simulations, which were run until well beyond the saturation time t_* listed in Table 2. In our numerical study, we adopt the usual definition of the driving wave number, $\lambda_d = 2\pi/k_{pk}$; and σ is defined as in section 1. While the quantity v_A specifies the initial condition of the fluid, the quantities σ, $\mathcal{M}$ and $\langle \dot{E} \rangle_t$ are obtained after saturation. In Table 2, the energies of the fluid are given by

$$E_k = \int \frac{1}{2} \rho v^2 \, d^3x, \tag{9}$$

$$\Delta E_B = \int \frac{B^2 - B_0^2}{2\,(4\pi)} \, d^3x = \int \frac{b^2 - b_0^2}{2} \, d^3x, \tag{10}$$

$$\Delta E_{th} = \int \rho \, c_s^2 \ln \left(\frac{\rho}{\rho_0} \right) \, d^3x, \tag{11}$$

and

$$\Delta E_{tot} = E_k + \Delta E_B + \Delta E_{th}. \tag{12}$$

Here, ΔE_{th} represents the compressional free energy (see Appendix). These results can be compared directly with the results of SOG and ML presented in Table 1.

Some numerical quantities of possible interest are listed in Table 3. In this table, $iter_*$ is the saturation iteration. For type II driving, ϕ is the constant normalization factor, and the quantity $\dot{\mathcal{E}}$ is the estimated energy injection rate when the fluid density is close to being uniform. Under such assumption, the estimated energy injection rate satisfies

$$\dot{\mathcal{E}} = \left(\frac{t_c}{\delta t} \right) \frac{M \phi^2}{2\,\delta t} \langle \delta v_1^2 \rangle \tag{13}$$

where $\langle\ \rangle$ denotes the expectation value over space and time, and $\boldsymbol{\delta v}_1$ is the velocity perturbation obtained when $\phi = 1$.

We obtain $\langle \delta v_1^2 \rangle$ by letting

$$\widetilde{\delta \boldsymbol{A}} = \sqrt{\left(\frac{L}{\Delta x} \right)^3}, \tag{14}$$

and carry out the remaining procedures for generating $\boldsymbol{\delta v}$, as in type II driving, with $\phi = 1$ and $t_c \to 0$. Performing this operation once, then taking the mean square of the resulting velocity field, we obtain the desired values for $\langle \delta v_1^2 \rangle$. In Table 4, we list the values of $\langle \delta v_1^2 \rangle$ as a function of k_{pk} and L/Δx.
[Lucy: See Table 2 and 3, check if $\mathcal{M} \propto \phi$.]

3.2 Turbulent energy spectrum

As a function of time and model.

[Lucy & Ue-Li: I've added below some "results" subsections on topics important to star formation theorists. Please take a look at Padoan & Nordlund (1999), Padoan & Nordlund (2002), and Krumholz & McKee (2005) to see examples of others' interest in this. – CDM]

3.3 Compressional density distribution function

3.4 Density-field strength relation

3.5 Collapsible cores: Population, formation rate, angular momentum

4 CONCLUSION

APPENDIX A: FREE ENERGY IN BAROTROPIC FLOW

Astrophysical flows are effectively barotropic ($P = P(\rho)$) if their radiative thermal time is short, allowing relaxation to a density-dependent equilibrium temperature that is rapid compared to dynamical density changes. The isothermal equation of state $P = \rho c_s^2$ used in the body of the paper is one example. In this Appendix we demonstrate that barotropic fluids possess a compressional free energy which is conserved in smooth flow and destroyed at shocks. Although minor, being $\mathcal{O}(\mathcal{M}^{-2})$ of the kinetic energy, we include it in the energy budget of turbulence.

In barotropic flow the conservation of mass

$$\partial_t \rho + \partial_i \rho v_i = 0 \tag{A1}$$

and momentum

$$\partial_t \rho v_i + \partial_j \left(\rho v_i v_j + p \delta_{ij} \right) = 0 \tag{A2}$$

suffice to determine the flow completely, whereas in adiabatic flow there is an additional energy equation

$$\partial_t e_f + \partial_i \left[(e_f + p) v_i \right] = 0. \tag{A3}$$

Here the free energy density is

$$e_f = \frac{1}{2} \rho v^2 + \rho \int^{1/\rho} p \, d\frac{1}{\rho} \tag{A4}$$

where the integral is along an adiabat if the gas is adiabatic. Note that $e_f = e$, the true energy density, in this case; for an isothermal gas, e_f is the sum of kinetic and Gibbs energy densities. We show that equations (A3) and (A4) remain valid for smooth barotropic flows as well as adiabatic ones. This is most easily shown in the Lagrangian form of the equations. Equation (A1) is equivalent to

$$\frac{\dot{\rho}}{\rho} + \nabla \cdot v = 0; \tag{A5}$$

equations (A1) and (A2) together give

$$\rho \dot{v} = -\nabla p. \tag{A6}$$

Equations (A5) and (A6) are equivalent to (A1) and (A2). Here the dot indicates the convective derivative: $\dot{f} \equiv \partial_t f + v \cdot \nabla f$.

Table 2. Our simulations: Numerical and physical parameters

	Numerical parameters			Physical parameters		Resolution		Equipartition		Dissipation rate	Saturation time
Model	$\frac{L}{\lambda_d}$	$\frac{\lambda_d}{\Delta x}$	$\frac{t_c \sigma}{\lambda_d}$	$\frac{v_A}{c_s}$	$\mathcal{M}$	$\Re$	$\Re_A$	$\frac{\Delta E_B}{\Delta E_{tot}}$	$\frac{\Delta E_{th}}{\Delta E_{tot}}$	$\frac{\lambda_d \langle \dot{E} \rangle_t}{M\sigma^3}$	$\frac{t_* \sigma}{\lambda_d}$
S8006410	8	64		10							
S8006403	8	64		3.2							
S8006401	8	64		1.0							
S8006400	8	64		0							
Z4025610	4	256		10		(2.56)					
Z4025607	4	256		7							
Z4025605	4	256		5		(10.24)					
Z40128xx	4	128		∞							
Z4012810	4	128		10	(10)	(1.28)					
Z4012807	4	128		7							
Z4012805	4	128		5	(5)	(5.12)					
Z4012800	4	128		0	(5)	(5.12)					
Z4006405	4	64		5	(5)	(2.56)					
Z4003205	4	32		5	(5)	(1.28)					
Z4001605	4	16		5	(5)	(0.64)					
Z4000805	4	8		5	(5)	(0.32)					
Z8006405	8	64		5	(5)	(2.56)					

Table 3. Our simulations: More numerical parameters

	Computing Resources		Saturation	Normalization	
Model	CPU number	CPU time	$iter_*$	ϕ	$\frac{\lambda_d \dot{\mathcal{E}}}{M\sigma^3}$
Z4012800	64				
Z4012805	64				
Z4012810	64				

The critical feature of the barotropic equation of state is the relation

$$d \int P d\frac{1}{\rho} = -P\frac{d\rho}{\rho^2}.$$

Applying this to equation (A4), we find for a barotropic gas that

$$\dot{e_f} = (e_f + p)\frac{\dot{\rho}}{\rho} + \rho v \cdot \dot{v}. \tag{A7}$$

We wish to know whether e_f is conserved, i.e., whether equation (A3) holds. This can be rewritten

$$\underbrace{\dot{e_f} - e_f\frac{\dot{\rho}}{\rho}}_{A} + \underbrace{v \cdot \nabla p}_{B} - \underbrace{p\frac{\dot{\rho}}{\rho}}_{C} = 0. \tag{A8}$$

¿From (A7),

$$A = p\frac{\dot{\rho}}{\rho} + \rho v \cdot \dot{v};$$

and from (A6),

$$B = -\rho v \cdot \dot{v}.$$

Therefore $A + B - C = 0$, proving (A3) for smooth (specifically, shock-free) barotropic flows.

A1 Free Energy Dissipation at Isothermal Shocks

Although free energy is conserved in barotropic flows without shocks, it is strictly dissipated across barotropic shocks. We demonstrate this for the isothermal case $p = \rho c^2$. In the shock frame, let the upstream density and velocity be ρ_0 and $v_0 = \mathcal{M}c$. The downstream density and velocity are then $\rho_1 = \mathcal{M}^2\rho_0$ and $v_1 = c/\mathcal{M}$. For the isothermal case,

$$e_f = \rho\left[\frac{1}{2}v^2 + c^2 \ln\left(\frac{\rho}{\rho_0}\right)\right] \tag{A9}$$

and the energy flux is

$$v(e_f + p) = \rho v\left[\frac{1}{2}v^2 + \left(1 + \ln\frac{\rho}{\rho_0}\right)\right].$$

Since mass conservation requires $\rho_0 v_0 = \rho_1 v_1$, the difference across the shock of the expression in square brackets determines the dissipation rate to be

$$-\left[v_0(e_f + p)\right]_0^1 = \rho_0 c^2 \mathcal{M}\left[\frac{1}{2}(\mathcal{M}^2 - \mathcal{M}^{-2}) - \ln\mathcal{M}^2\right]. \tag{A10}$$

This function is strictly positive for $\mathcal{M} > 1$, implying that free energy is strictly dissipated. For $\mathcal{M} \gg 1$, the rate of dissipation asymptotes to the upstream free energy flux.

REFERENCES

Arons J., Max C. E., 1975, ApJL, 196, L77
Bertoldi F., McKee C. F., 1992, ApJ, 395, 140
Crutcher R. M., 1999, ApJ, 520, 706
Gardiner T. A., Stone J. M., 2005, Journal of Computational Physics, 205, 509
Heitsch F., Mac Low M., Klessen R. S., 2001, ApJ, 547, 280

Table 4. Values of $\langle \delta v_1^2 \rangle / c_s$

k_{pk}	Ł/Δx					
	32	64	128	256	512	1024
4	0.076815369	0.56421394	4.4131727	35.086071	278.58064	
8						

Hirsch C., 1990, Numerical Computation of Internal and External Flows, vol. 2: Computational Methods for Inviscid and Viscous Flows. Publisher: New York, US: John Wiley, 1990

Johnstone D., Di Francesco J., Kirk H., 2004, ApJL, 611, L45

Klessen R. S., Heitsch F., Mac Low M., 2000, ApJ, 535, 887

Krumholz M. R., McKee C. F., 2005, ArXiv Astrophysics e-prints

Larson R. B., 1981, MNRAS, 194, 809

Lazarian A., Vishniac E. T., 1999, ApJ, 517, 700

Mac Low M.-M., 1999, ApJ, 524, 169

Matzner C. D., 2002, ApJ, 566, 302

McKee C. F., 1989, ApJ, 345, 782

Norman C., Silk J., 1980, ApJ, 238, 158

Onishi T., Mizuno A., Kawamura A., Ogawa H., Fukui Y., 1998, ApJ, 502, 296

Padoan P., Nordlund Å., 1999, ApJ, 526, 279

Padoan P., Nordlund Å., 2002, ApJ, 576, 870

Padoan P., Zweibel E., Nordlund Å., 2000, ApJ, 540, 332

Pen U., Arras P., Wong S., 2003, ApJS, 149, 447

Solomon P. M., Rivolo A. R., Barrett J., Yahil A., 1987, ApJ, 319, 730

Stone J. M., Ostriker E. C., Gammie C. F., 1998, ApJL, 508, L99

Tielens A. G. G. M., Hollenbach D., 1985, ApJ, 291, 722

This paper has been typeset from a TEX/ LATEX file prepared by the author.