

IQI Discussion: $(2+1)D$ Extended TQFTs

MATH

April 22, 20



Doubled Topological Phases

Theories of Excitations

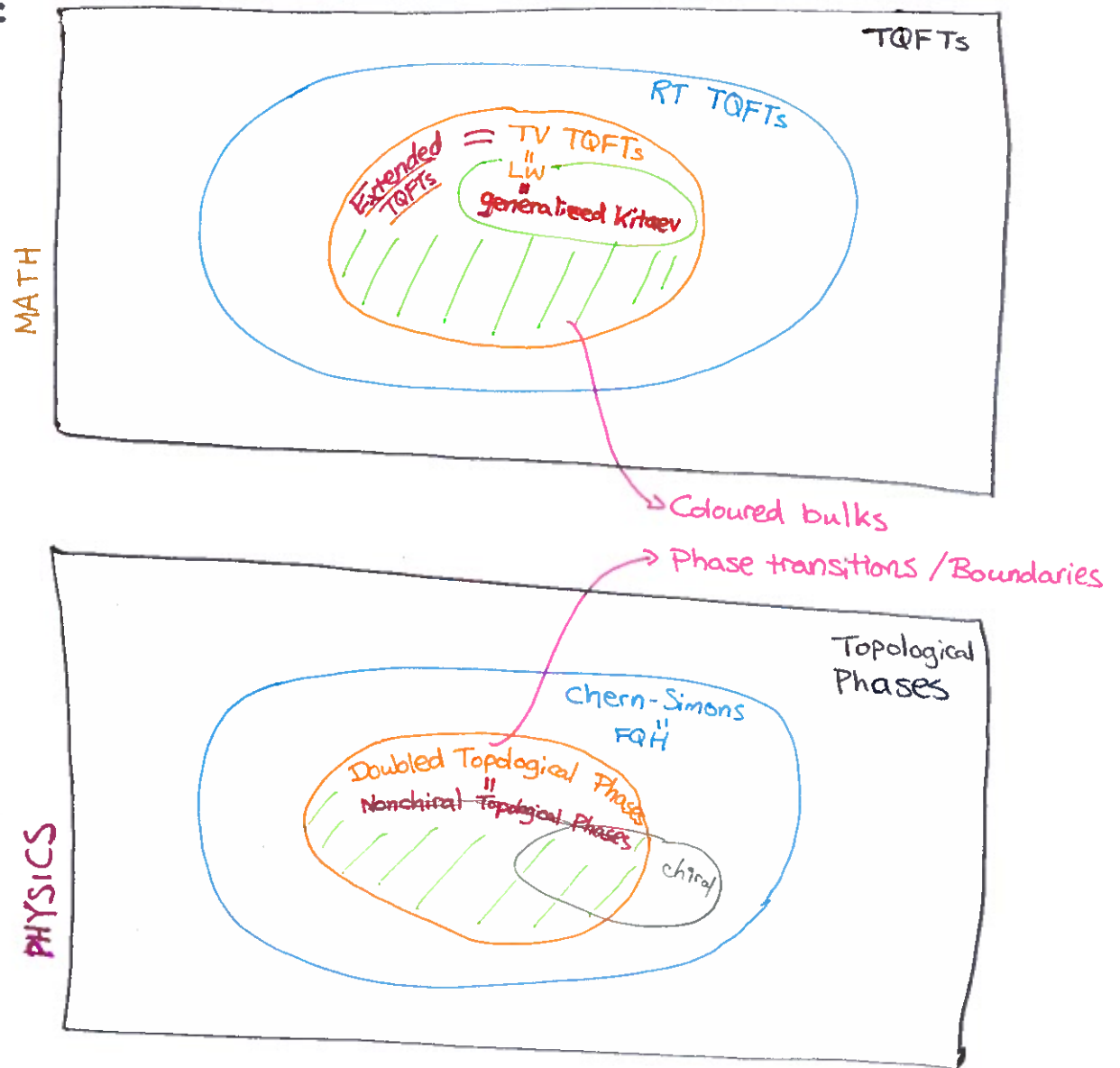
PHYSICS

- Goals:
- ① Gain understanding of coloured, extended TQFTs
 - 3 Cob
 - 3-categories.

- ② Raise questions about physical theory vs. mathematical theory.
 - some progress / guesses

Comments: Interrupt anytime to ask math/physics questions.

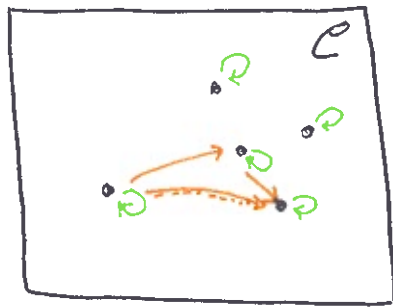
Landscapes:
in $(2+1)D$



(2+1)D TQFTs (with an increasing amount of structures)

General idea: (of TQFTs)
 "Z": "3 Cob" \rightarrow "Alg"
 ↑ Some functor ↑ some topological category (Cobordisms) ↑ some algebraic category

Step 1: Categories & Functors

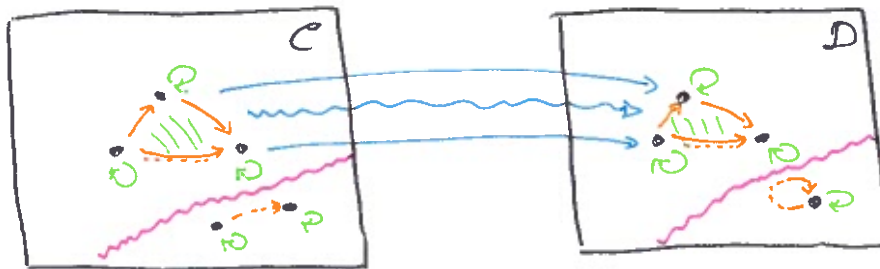


Category C:

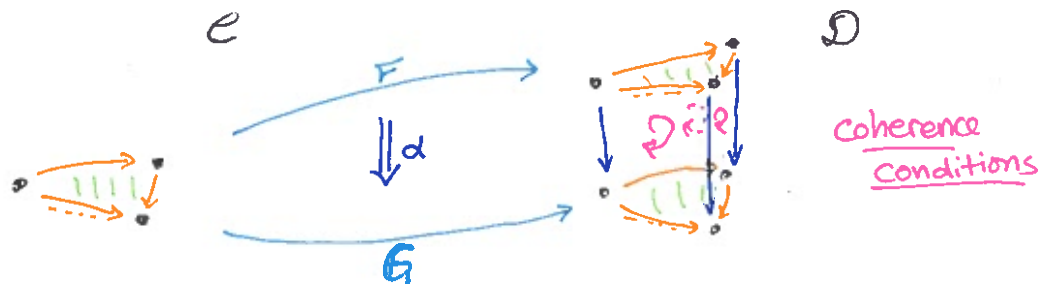
data Objects A, B, \dots
 Morphisms f, g, \dots Source & Target $\text{Hom}(A, B)$

Axioms ① Composition
 ② Identity

Functor $F: C \rightarrow D$ maps objects & morphisms



Natural Transformation $\alpha: F \rightarrow G$ maps in(Objects) in D only



Observation: A 1-category resembles a 1-dim'd CW-complex.

Physics: Often we are dealing with concrete categories,

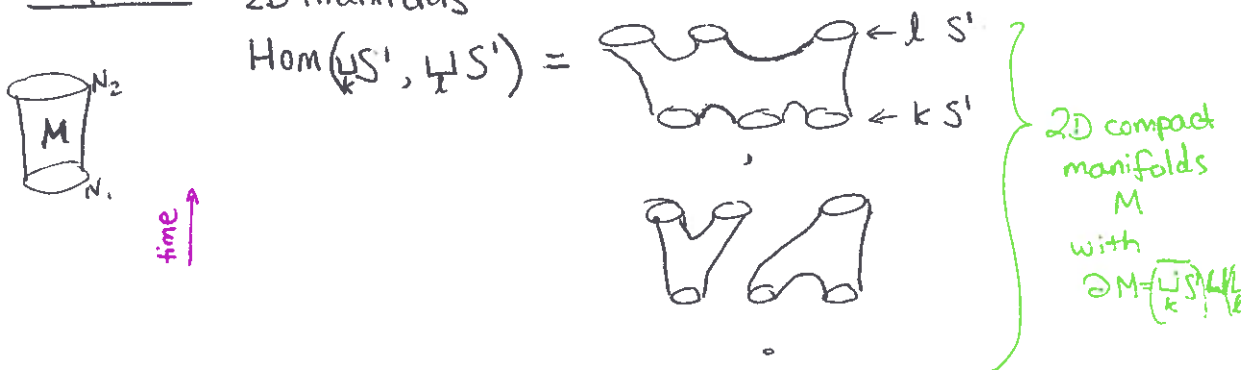
Step 2 : (1+1)D TQFTs

- Need to describe functor

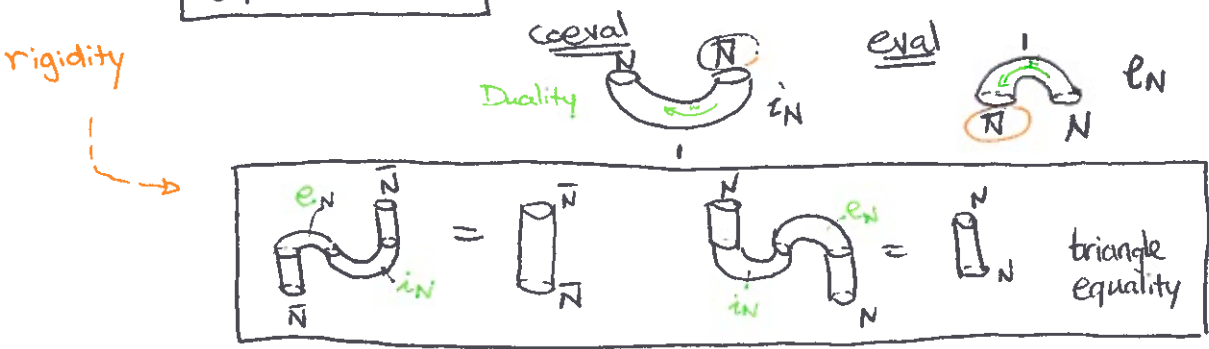
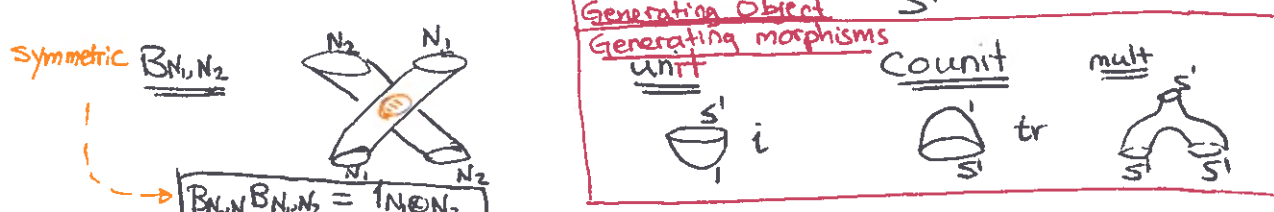
$$\mathbb{Z}_2: 2\text{Cob} \longrightarrow \text{Vect} \quad \left(\text{Such a functor is call a } \underline{\underline{2D TQFT}} \right)$$

First, describe the categories

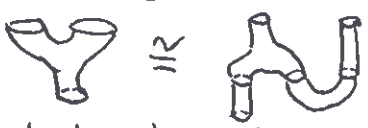
- 2Cob: \leftarrow rigid symmetric monoidal category monoidal unit either orientation
 - Objects: 1D manifolds, i.e. $\emptyset, \text{circle}, \text{two circles}, \text{three circles}, \dots$
 - Morphisms: 2D compact manifolds



Equivalences between objects: via homeomorphisms of 1-manifolds



Eg(of generating morphisms):



special kind of v.sp.

- Vect: repeat the above story. \rightsquigarrow need A to make all the above work.

Step 2: (1+1)D TQFTs (cont'd)

• Let's examine $Z_2: 2Cob \rightarrow Vect$, $\emptyset \mapsto V$, $\text{cup} \mapsto \text{Hom}(V, V)$

Since both LHS & RHS categories have rigid, symmetric, monoidal structure, we demand that Z_2 preserve these structures.

Call this functor Z_2 (in a box)

$\emptyset \mapsto A$ (a f.d. commutative Frobenius algebra)
 equipped with $\epsilon: A \rightarrow \mathbb{C}$ s.t. $(a, b) = \epsilon(a \cdot b)$ nondegenerate
 $\text{cup} \mapsto \mu: A \otimes A \rightarrow A$
 $\text{cap} \mapsto \epsilon: A \rightarrow \mathbb{C}$
 $\text{cylinder} \mapsto \eta: \mathbb{C} \rightarrow A$

Step 3: (2+1)D TQFTs

A (2+1)D TQFT is a functor

$Z_3: 3Cob \rightarrow Vect$

• 3Cob:

Objects 2D manifolds with $\partial=0$, ie. genus g surfaces (compact, oriented)

Morphisms 3D manifolds (compact oriented); $\text{Hom}(N_1, N_2) = \{M \text{ 3-manifold} \mid \partial M = \bar{N}_1 \cup N_2\}$

Structures similar to 2Cob.

Remark

Reshetikhin - Turaev construction

with input a modular (monoidal) category: $\otimes, *, \text{braiding, mod}$ produces a family of 3-2-1 TQFTs (not-0)



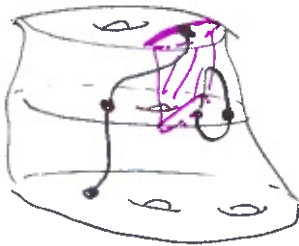
e.g. $e = \text{Rep } U_g$, $q = e^{2\pi i/x}$
 $Z_{RT}^e(\emptyset) = \mathbb{C}$
 $Z_{RT}^e(\text{torus}) = e \times e \times e \rightarrow Vect$

Step 4: (Fully) Extended (2+1)D TQFTs

A (fully) extended (2+1)D is a 3-functor

$$\mathbb{Z}_4: 3\text{Cob}^{\text{ext}} \longrightarrow \text{Mon}$$

- 3Cob^{ext}



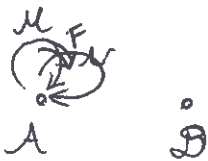
objects 0-dim topological features

1-morph 1-dim'l, e.g. worldlines of particles

2-morph 2-dim'l, e.g. time slice, transparent domain wall

3-morph 3-dim'l e.g. spacetime

- Mon



objects monoidal categories $\mathcal{A}, \mathcal{B}, \dots$

1-morph $\mathcal{A} - \mathcal{A}$ bimodule categories \mathcal{M}

(ie. \mathcal{M} with structure maps
 $\mathcal{A} \otimes \mathcal{M} \rightarrow \mathcal{M}$
 $\mathcal{M} \otimes \mathcal{A} \rightarrow \mathcal{M}$)

2-morph $\mathcal{M} \rightarrow \mathcal{N}$ bimodule functors \mathcal{F}

3-morph functorial isomorphism $\phi: \mathcal{F} \rightarrow \mathcal{G}$ (natural)

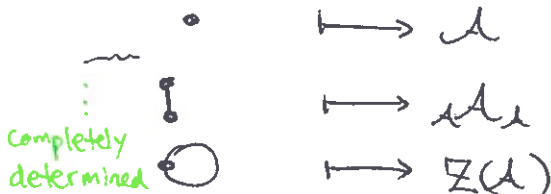
- 3-functor ...

Remark Turaev-Viro Construction

with input a spherical (monoidal) category: $\otimes, *$

produces a family of 3-2-1-0 TQFTs (fully extended)

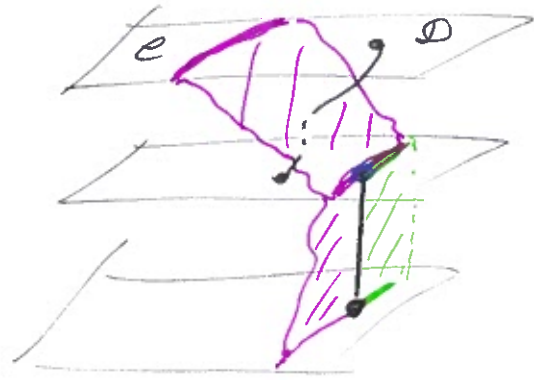
$$\mathbb{Z}_{TV}^{\mathcal{A}}: 3\text{Cob}^{\text{ext}} \longrightarrow \text{Mon.}$$



Conjecture:

3D Extended TQFTs \leftrightarrow Spherical categories

Step 5: Coloured, Extended (2+1)D TQFTs



(beyond Lurie ~2009)

3-functor

$$\mathbb{Z}_5: \text{3Cob}^{\text{ext. coloured}} \rightarrow \text{Bimod.}$$

update

bulks coloured

update

[Kitaev & Kong, Kirillov]

Morita equivalence ($\mathbb{Z}(e) = \mathbb{Z}(\emptyset)$)

Spatial phase transition (Domain walls)

... Discussions ...

Colours vs. Invariants

In Kitaev & Kong,

(bulk) 3-dim'l	\rightsquigarrow	<u>colours</u> monoidal category \mathcal{A}, \mathcal{B}
(time slices) (domain wall) 2-dim'l	\rightsquigarrow	\mathcal{A} - \mathcal{B} bimodule category
(e.g. twists) 1-dim'l	\rightsquigarrow	bimodule functor $F: \mathcal{M} \rightarrow \mathcal{N}$
(instanton) 0-dim'l	\rightsquigarrow	natural isom. $\phi: F \rightarrow G$.

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