AdS/CFT and quantum gravity or What is AdS/CFT?

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2008

hep-th/0804.0632

Background: what is AdS/CFT

AdS/CFT has a very long history and correspond to a vast litterature > 2000 papers

However only a few portion of the litterature is really adressing the question relevant quantum gravity community worried about background independence

Witten	Original formulation useful for this and still the best ref
Skenderis	A key semi-classical understanding of the correspondance and “holographic renormalisation group”
De Boer, verlinde	Some claim about the hamilton-jacobi gravity equation and renormalisation group equation
maldacena	Appendix of a paper in non-gaussianity and dS/CFT picture

What is the purpose of this talk?

My original motivation is related to some recent proposal in 3d:
A proposal for a CFT “defining” 3d quantum gravity

However we a priori have an independent definition of what 3D quantum gravity is. Independentely of any CFT definition

Can we prove/disprove AdS/CFT?

several points to adress:

Is there a formulation of AdS/CFT that allows us to prove/disprove it:
What is the exact dictionary between Quantum gravity and CFT?

Is the correspondance between one theory of QG and one CFT or one to many?

Is there one CFT associated to QG or some vaccuum sector of it?

Is there a background independent formulation of AdS/CFT

Do we care if we are only interested in quantum gravity, why?

Can we reconstruct quantum gravity from a Boundary CFT?

I want to foster discussions on his subject and propose a precise dictionnary, eventually a bulk reconstruction formula.

Basics classical basis of AdS/CFT

Lets look at a d+1 dimensional spacetime M,g solution of Einstein with cc

$$R_{\mu\nu}(g) = -\epsilon \frac{d}{\ell^2} g_{\mu\nu}, \qquad \Lambda = -\epsilon \frac{d(d-1)}{2\ell^2}$$

$\epsilon = -1$ for dS $\quad +1$ for AdS.

M is conformally compact if there exists a defining function

$$\rho^{-1}(0) = \partial M \text{ and } d\rho \neq 0$$
$$\ell^2 \bar{g} = \rho^2 g$$

such that the conformally equivalent metric extend smoothly to ∂M

The einstein equation imply that $\quad \bar{g}^{\mu\nu}\partial_\mu \rho \partial_\nu \rho = \epsilon \quad$ on ∂M

Fefferman-Graham expansion: we can always chose ρ such that this is true in a neighborood of ∂M $\quad \ell\rho(x) = d_{\bar{g}}(\partial M, x)$

$$ds^2 = \frac{\ell^2}{\rho^2}(\epsilon d\rho^2 + \gamma_\rho), \quad ds^2 = (\epsilon dr^2 + \ell^2 e^{\frac{2r}{\ell}} \gamma_r), \quad \rho = \exp\left(-\frac{r}{\ell}\right)$$

Classical basis of AdS/CFT

In a neighborood of ∂M

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$\rho^{-2}\gamma_\rho$ is the metric induced on the surfaces Σ_ρ with $\rho = cste$

Σ_ρ are spacelike in the dS case and timelike in AdS

A key property of the extrinsic curvature tensor $K^\nu_\mu = \frac{1}{2}h^{\nu\alpha}\mathcal{L}_n g_{\alpha\mu}$

$$\ell K^j_i = \left(\delta^j_i - \rho(\gamma^{-1}\partial_\rho\gamma)^j_i\right) = \delta^j_i + O(\rho^2).$$

Provide the heuristic basis for the AdS/CFT correspondance

$$\partial_r\Psi(\gamma_\rho) = \int_{\Sigma_\rho}\partial_r\gamma_{ij}\frac{\delta\Psi}{\delta\gamma_{ij}} \sim_{\rho=0} \frac{2}{\ell}\int_{\Sigma_0}\gamma_{ij}\frac{\delta\Psi}{\delta\gamma_{ij}}.$$

Due to cc the radial evolution is equivalent near infinity to conf rescaling

Infinity is left invariant by radial evolution $\longrightarrow$ conformal invariance of the physics describe by Ψ at asymptotic infinity

AdS/CFT the original formulation

AdS/CFT is a equivalence between the quantum gravity partition function with fixed Dirichlet boundary condition and the generating functional of connected correlation function of CFT theory

namely lets $\Phi_i = \Phi, g_{\mu\nu}, \ldots$ Bulk (scalar, gravitons,...) fields

Bulk side: chose an Asymptotic AdS spacetime and define the amplitude

$$\Psi_{\Sigma_\rho}(\phi_i) = \int_{\Phi_i|_{\Sigma_\rho}=\phi_i} \mathrm{D}\Phi_i \; e^{iS_{B,M}(\Phi_i)}$$

Faddev, slavnov 75

(Ad)S-matrix functional

CFT One associate conformal operators of dimension Δ_i

$\gamma \rightarrow \rho^{-2}\gamma$ the fields transform as $\phi_i \rightarrow \rho^{d-\Delta_i}\phi_i$.

$$Z_{CFT}(\phi_i) = \langle e^{\int_\Sigma \phi_i \hat{O}_i} \rangle$$

AdS/CFT : there exist a CFT such that we have the equality

$$\Psi_{\Sigma_0}(\phi_i) = Z_{CFT}(\phi_i).$$

Three puzzles

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1 Technical puzzle:

The LHS is evaluated at asymptotic infinity Divergences arise even at the classical level that needs to be regularised

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The equivalence refers to a background asymptotic spacetime and its slicing. In quantum gravity the metric is dynamical and information about the bulk geometry should be encoded into the choice of Ψ. But if so where is asymptotic infinity?

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3 Equations : The LHS and RHS do not satisfy the same equations!

The LHS satisfy Wheeler-de-Witt equation which is second order diff eq.

The RHS satisfies a conformal Ward identity which is first order.

How can we have equivalence between solutions of a first and second order differential system!!

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resolution of these puzzles is a key to the understanding of AdS/CFT

Holographic renormalisation group

Gravity equations

metric γ on a d dimensional space Σ

boundary of a d+1 dimensional manifold M, $\partial M = \Sigma$

$$\Psi_\Sigma(\gamma) = \int_{g|_{\partial M}=\gamma} \mathrm{D}g \; e^{iS_M(g)}$$

$$S_M(g) = -\left(\frac{1}{2\kappa}\int_M \mathrm{d}^{d+1}x\sqrt{|g|}\,(R(g)-2\Lambda) + \frac{\epsilon}{\kappa}\int_\Sigma \mathrm{d}^d x\sqrt{|\gamma|}K\right)$$

$$\delta S = \frac{1}{2\kappa}\int_\Sigma \sqrt{\gamma}\,\Pi^{ab}\delta\gamma_{ab}, \qquad \Pi^{ab} = \epsilon\left(K^{ab} - \gamma^{ab}K\right)$$

Under bulk diffeomorphism non vanishing at the boundary $\quad \xi_n = \xi^\mu n_\mu$

$$\begin{aligned}\delta_\xi S_M(g) &= -\frac{1}{2\kappa}\int_{\partial M} \mathrm{d}^d x \xi_n\left(\sqrt{\gamma}\,(R(g)-2\Lambda) + 2\mathcal{L}_n(\sqrt{|\gamma|}K)\right)\\ &= -\frac{1}{2\kappa}\int_{\partial M} \mathrm{d}^d x\sqrt{\gamma}\xi_n\left(R(\gamma) - 2\Lambda + \epsilon(K^2 - K^b_a K^a_b)\right)\end{aligned}$$

Gravity equations

$$\Psi_\Sigma(\gamma) = \int_{g|_{\partial M}=\gamma} \mathrm{D}g\ e^{iS_M(g)}$$

Under bulk diffeomorphism non vanishing at the boundary: radial Wheeler-de-Witt equation

$$\hat{\Pi}^{ab}_x \equiv \frac{2}{\sqrt{\gamma}} \frac{\delta}{\delta\gamma_{ab}(x)}, \quad \hat{\Pi}^a_b \equiv \gamma^{ac}\hat{\Pi}_{cb}, \quad \hat{\Pi} \equiv \hat{\Pi}^c_c$$

$$\mathcal{H}\Psi_\Sigma = \mathcal{H}_a\Psi_\Sigma = 0$$

Where

$$\kappa = 8\pi G$$

$$\mathcal{H}_b = \nabla_a \hat{\Pi}^a_b$$

$$\mathcal{H} = -\epsilon\kappa^2 : \left(\hat{\Pi}^b_a\hat{\Pi}^b_a - \frac{\hat{\Pi}^2}{d-1}\right) : + R(\gamma) + \epsilon\frac{d(d-1)}{l^2}$$

$: :$ denotes renormalisation that can be carried out explicitely at any loop order by substracting the singular part of the boundary to boundary propagator $\mathcal{K}_{\mathcal{S}\,ab}^{\ ab}(x,x)$

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Gravity equations

$$\Psi_\Sigma(\gamma) = \int_{g|_{\partial M}=\gamma} \mathrm{D}g \; e^{iS_M(g)}$$

Under bulk diffeomorphism non vanishing at the boundary:
radial Wheeler-de-Witt equation

$$\gamma = e^{2\phi}\hat{\gamma}, \; \det(\hat{\gamma}) = 1 \qquad e^{-d\phi}\hat{P}^a_b \equiv \hat{\Pi}^a_b - \frac{\delta^a_b}{d}\hat{\Pi}$$

acts on $\hat{\gamma}$ only preserves its unimodularity and commute with $\delta/\delta\phi$.

$$\mathcal{H} = \frac{\kappa^2}{d(d-1)}\left(\frac{\delta}{\delta\phi}\right)^2 - \kappa^2 e^{-2d\phi}\hat{P}^2 + e^{-2\phi}R(\hat{\gamma}) - 2(d-1)\left(\hat{\Box}\phi + (d-2)(\hat{\nabla}\phi)^2\right) + \frac{d(d-1)}{l^2}$$

A relativistic equation $\longrightarrow$ One expect existence of non trivial kernels $K_\phi, \tilde{K}_\phi$ such that

$$\Psi(e^{2\phi}\hat{\gamma}) = \int \mathcal{D}\hat{\gamma}' \left(K_\phi(\hat{\gamma},\hat{\gamma}')\Psi(\hat{\gamma}') + \tilde{K}_\phi(\hat{\gamma},\hat{\gamma}')\frac{\delta\Psi}{\delta\phi}(\hat{\gamma}')\right)$$

“Initial data” determined by $\Psi(\hat{\gamma})$ $\quad \frac{\delta}{\delta\phi}\Psi(e^{2\phi}\hat{\gamma})|_{\phi=0}$

CFT equations

A CFT partition function is by definition a solution of two equations diffeomorphism constraints and a conformal Ward identity

$$\nabla_a \hat{\Pi}^a_b Z_{CFT}(\gamma) = 0$$

$$\frac{1}{\sqrt{\gamma}} \frac{\delta}{\delta \phi(x)} Z_{CFT}(e^{2\phi}\gamma)\Big|_{\phi=0} = i A_d(x) Z_{CFT}(\gamma)$$

Invariance under local rescaling

Anomaly

$$A_d(x) = 0$$ For d odd

a,c central charges

$$A_2(x) = \frac{c}{12\pi} R(x)$$

$$A_4(x) = \frac{1}{16\pi}\left(aE(x) - cW^2(x) + \alpha \Box R(x)\right)$$

+ modular invariance

$$E = \left(\frac{1}{2}\epsilon^{ef}_{ab} R_{efcd}\right)^2$$ Euler density

Weyl square

Resolution of the puzzles

Lets start by adressing the problem of background independence

The key point is to remember that at the classical level the existence and the property of asymptotic infinity follows dynamically from the einstein equation and the assumption that the limit $\rho \to 0$ exists

$$\bar{g} = \rho^2 g$$

At the quantum level the quantum spacetime is represented by $\Psi_\Sigma(\gamma)$

Where is asymptotic infinity ?

If we remember that at the classical level the induced metric scales as $\frac{\gamma}{\rho^2}$

One needs to look at the behavior of $\Psi\left(\frac{\gamma}{\rho^2}\right)_{\rho \to 0}$ for a solution of WdW

rescaling in ρ correspond to a radial motion of the slice Σ

asymptotic property of $\Psi_\Sigma(\gamma)$: asymptotic property of the semi-classical spacetime it represents

Asymptotic behavior of WdW solution

A key lemma:

Let $\Psi_\Sigma(\gamma)$ a solution of radial WdW equation then the asymptotic of

a solution of radial WdW when $\rho \to 0$ is given by

$$\Psi\left(\frac{\gamma}{\rho^2}\right) \sim e^{\frac{i}{\kappa}S^{(d)}_{loc}\left(\frac{\gamma}{\rho^2}\right)} Z_+(\gamma) + e^{-\frac{i}{\kappa}S^{(d)}_{loc}\left(\frac{\gamma}{\rho^2}\right)} Z_-(\gamma)$$

$S^{(d)}_{loc}(\gamma)$ is an explicit local action containing terms of dimension at most d/2

$Z_\pm(\gamma)$ are a pair of CFT's: Solutions of Ward identity

$$\frac{1}{\sqrt{\gamma}} \frac{\delta}{\delta\phi(x)} Z_{CFT}(e^{2\phi}\gamma)\Big|_{\phi=0} = iA_d(x) Z_{CFT}(\gamma)$$

Which is a left-over (holographic in-print) of the Wheeler de Witt equation

Born-oppeiheimer expansion where ∞ is the heavy component

Asymptotic behavior of WdW solution

$$\Psi\left(\frac{\gamma}{\rho^2}\right) \sim e^{\frac{i}{\kappa}S^{(d)}_{loc}\left(\frac{\gamma}{\rho^2}\right)} Z_+(\gamma) + e^{-\frac{i}{\kappa}S^{(d)}_{loc}\left(\frac{\gamma}{\rho^2}\right)} Z_-(\gamma)$$

What does that mean?

A general quantum gravity state correspond to a pair of CFT's:not to one

If one look at the extrinsic curvature $\frac{\kappa}{i}\hat{\prod}^{ab}\Psi$ the two terms correspond to $\ell K_i^j = \pm\delta_i^j + O(\rho^2)$

One of the two corresponds to reaching AS from the "inside" the other one from the "outside"

A state corresponding to a spacetime with AS ∞ should be such that $Z_-(\gamma) = 0$

This is our definition of radial "states" describing AS ∞

For such states: The Boundary CFT is the initial data on the slice Σ_0 at ∞ determining the gravity state

The correspondance is NOT one to one: It is a correspondence between solution of QG (radial "states") and different CFT Theories

Asymptotic behavior of WdW solution

$$\Psi\left(\frac{\gamma}{\rho^2}\right) \sim e^{\frac{i}{\kappa}S^{(d)}_{loc}\left(\frac{\gamma}{\rho^2}\right)} Z_+(\gamma) + e^{-\frac{i}{\kappa}S^{(d)}_{loc}\left(\frac{\gamma}{\rho^2}\right)} Z_-(\gamma)$$

This resolves the 3 puzzles

1 Technical puzzle: The infinities are exactly substracted by $S^{(d)}_{loc}(\gamma)$

2 background free: AS ∞ is in the asymptotic behavior of $\Psi_\Sigma(\gamma)$

3 equations: The CFT Ward identity is the in-print of WdW at ∞

1st versus 2nd order: 1 gravity solution generically corresponds to 2 CFT's

Asymptotic behavior of WdW solution

$$\Psi\left(\frac{\gamma}{\rho^2}\right) \sim e^{\frac{i}{\kappa}S_{loc}^{(d)}\left(\frac{\gamma}{\rho^2}\right)} Z_+(\gamma) + e^{-\frac{i}{\kappa}S_{loc}^{(d)}\left(\frac{\gamma}{\rho^2}\right)} Z_-(\gamma)$$

What does it mean then to identify THE CFT dual to gravity

e-g SYM in d =4+1 or Witten Proposal in d=2+1

What is meant is to identify THE CFT corresponding to a particular gravity state, the vaccua state which is supposed to be

$$\Psi_H(\gamma) = \int_{g|_{\partial H}=[\gamma]} e^{\frac{i}{\kappa}S_H(\gamma)(g)}$$

Where H is the handlebody associated to Σ

eg in d=2 Σ H is the plain torus

Still to be done in 3d

Proof

$$\Psi\left(\frac{\gamma}{\rho^2}\right) \sim e^{\frac{i}{\kappa}S^{(d)}_{loc}\left(\frac{\gamma}{\rho^2}\right)} Z_+(\gamma) + e^{-\frac{i}{\kappa}S^{(d)}_{loc}\left(\frac{\gamma}{\rho^2}\right)} Z_-(\gamma)$$

$$S^{(d)}_{loc}(\gamma) = \frac{(d-1)}{\ell}\int_\Sigma \sqrt{\gamma} + \frac{\ell}{2(d-2)}\int_\Sigma \sqrt{\gamma}R(\gamma) + \frac{\ell^3}{2(d-4)}\int_\Sigma \sqrt{\gamma}\left(P^b_a P^a_b - PP\right)$$

$$\frac{\ell}{(d-2)}\left(R^b_a - \frac{R}{2(d-1)}\delta^b_a\right) \equiv \ell P^b_a$$

+ loop correction renormalise the coeff

in d=2,3 only 1-loop correction matters

No obvious sign of non-renormalisability

in d=4,5 only 2-loop correction matters ...

Terms of dimension > d vanish in the limit of $S^{(d)}_{loc}\left(\frac{\gamma}{\rho^2}\right)$

One has to had to $S^{(d)}_{loc}\left(\frac{\gamma}{\rho^2}\right)$ In d even a term $\ln\rho\, A_d$

Proof

One can rescale the WdW constraint

$$\gamma_{ab} \longrightarrow \rho^{-2}\gamma_{ab} \quad \hat{\Pi}^a_b \longrightarrow \rho^{-d}\frac{2}{\sqrt{\gamma}}\gamma^{ac}\frac{\delta}{\delta\gamma_{cb}} \equiv \rho^{-d}\hat{\Pi}^a_b$$

Then $\Psi(\rho^{-2}\gamma) \equiv \Psi_\rho(\gamma)$ is a solution of $\mathcal{H}_\rho\Psi_\rho(\gamma) = 0$ with

$$\mathcal{H}_\rho = -\epsilon\kappa^2\rho^{2d}\,\hat{\Pi}\cdot\hat{\Pi} + \epsilon\frac{d(d-1)}{l^2} + \rho^2 R(\gamma)$$

in the limit $\rho \to 0$

$$\Psi^{(0)}_\rho(\gamma) = \exp\left(\pm\frac{i}{\kappa\rho^d}\frac{d-1}{\ell}\int_\Sigma\sqrt{\gamma}\right)$$

is a solution of

$$\kappa^2\rho^{2d}\,\hat{\Pi}\cdot\hat{\Pi}\Psi^{(0)}_\rho(\gamma) = \frac{d(d-1)}{l^2}\Psi^{(0)}_\rho(\gamma).$$

Proof

One can expand around

$$\Psi^{(0)}_{\rho}(\gamma) = \exp\left(\pm\frac{i}{\kappa\rho^d}\frac{d-1}{\ell}\int_{\Sigma}\sqrt{\gamma}\right) \qquad \Psi \; = \; \Psi^{(0)}\Psi^{(1)}$$

Since $$\rho^d\hat{\Pi}^a_b\Psi \; = \; \Psi^{(0)}\left(i\tfrac{d-1}{\kappa\ell}\delta^a_b + \rho^d\hat{\Pi}^a_b\right)\Psi^{(1)}$$

rWdW becomes

$$i\frac{2\kappa}{\ell}\rho^d\hat{\Pi}\Psi^{(1)} + \rho^2 R(\gamma)\Psi^{(1)} - \kappa^2\rho^{2d}\,\hat{\Pi}\circ\hat{\Pi}\Psi^{(1)} = 0$$

Thus for d=2

$$\hat{\Pi}\Psi^{(1)} = i\frac{\ell}{2\kappa}R(\gamma)\Psi^{(1)}$$

etc...

A reconstruction formula

If The CFT is just some initial data associated to a surface at infty then one should be able to reconstruct the bulk from the CFT via a propagating kernel.

Can we do it ?

A reconstruction formula

If The CFT is just some initial data associated to a surface at infty then one should be able to reconstruct the bulk from the CFT via a propagating kernel.

Can we do it ? In 3d yes explicitely

A reconstruction formula

If The CFT is just some initial data associated to a surface at infty then one should be able to reconstruct the bulk from the CFT via a propagating kernel.

Another Lemma: Given Any 2d CFT Z_c with central charge c

We can construct a solution of WdW equation with asymptotic Z_c

$$ds^2 = e^+ e^-$$

$$\Psi\left(e^{\pm}\right) = \exp\left(\frac{ik}{2\pi}\int_\Sigma e\right)\int DE\, \exp\left(-\frac{ik}{\pi}\int_\Sigma (E^+ - e^+)\wedge(E^- - e^-)\right) Z_c(E)$$

as long as

$$c = 1 + 6k \qquad\qquad k = \frac{\ell}{4G}$$

Amazingly simple formula!

Very long proof...

A reconstruction formula

$$Z_c$$

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Conclusion

The gravity AdS matrix satisfies the radial WdW equation

The asymptotic value of any solution is control by a pair of CFT

Restricting to one CFT is equivalent to looking at spacetime with AS ∞

The correspondence is one to many that is
between radials states of gravity and boundary CFTs

We can give in 2+1 an explicit reconstruction formula of a radial state
from any boundary CFT

We have proposed a particular radial state to study in order to
identify the CFT dual to gravity
many open questions:
Lorentzian vs Euclidean? Relationship between radial states and
usual states or more density matrix of QG.
dSitter vs AdS, meaning of the boundary imaginary CFT....