Problem Set #7

Quantum Error Correction
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Due Tues., March 6, 2007

Problem #1. Concatenated Codes

For parts a and b, consider concatenating two codes with stabilizers S_1 and S_2. S_1 is an $[[n_1, k_1, d_1]]$ code, and we concatenate by encoding each physical qubit of S_1 in S_2, which is an $[[n_2, 1, d_2]]$ code. (Note that the second code encodes only $k_2 = 1$ qubit.)

a) Determine a set of generators for the stabilizer of the concatenated code in terms of the generators and logical Pauli operations of S_1 and S_2.

b) Determine the parameters $[[n, k, d]]$ of the concatenated code.

c) Suppose now that S_2 is an $[[n_2, k_2, d_2]]$ code with $k_2 > 1$, and that $n_1 = ak_2$. Imagine we concatenate by dividing the physical qubits of S_1 into a blocks of size k_2, and encoding those blocks into S_2. Find the parameters $[[n, k, d]]$ of the resulting concatenated code (assuming no special properties for S_1 or S_2). Can you figure out a concatenation method that gives a better distance when S_1 is a code over higher-dimensional qudits?

Problem #2. Threshold for Classical Reversible Computation

The techniques we used in class to prove the threshold theorem for quantum computation can be applied directly to classical computation as well. The only major difference (besides needing a different universal gate set) is that there is no measurement; instead, let us imagine ending the computation with an ideal decoder, which could perhaps represent the experimenter looking at the outcome of the computation and decoding it himself.

In this problem, we will use those techniques to derive a lower bound on the threshold for classical reversible computation using the concatenated 3-bit repetition code ($\overline{0} = 000$, $\overline{1} = 111$). Assume below that preparing a bit either as 0 or 1, and performing a NOT gate, a CNOT gate, and a Toffoli gate can each be done in 1 time step and with error rate p. (This applies to the physical gates, not the logical ones.) Similarly, a bit that waits for one time step without a gate undergoes a storage error with probability p. Other assumptions are as in the quantum case.

a) The Toffoli gate, which maps $(a, b, c) \mapsto (a, b, c \oplus ab)$, is universal for classical reversible computation along with preparation of ancilla bits in the state 0 or 1. Note that the Toffoli gate and 0 and 1 ancilla preparation can all be done transversally on the repetition code. Count the number of locations in the preparation extended rectangle and the Toffoli extended rectangle in terms of the number of locations in an error correction step.

b) Design a fault-tolerant error correction circuit for the 3-bit repetition code; that is, one that satisfies EC properties 1 and 2. (Remember that you have no measurement available, and must use noisy gates to analyze the syndrome and determine the error that needs to be corrected.) Count the number of locations in your error correction circuit.

c) Use the techniques from class to determine a lower bound on the threshold for the concatenated 3-bit repetition code.