Quantum Error Correction

Notes for lecture 1

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Basic problem : Let $\mathbf{S}$ be a superoperator representing the error(s) that can occur during the manipulation (or transmission) of a qubit.

$$\rho \longrightarrow \mathbf{S}[\rho] \xrightarrow{?} \rho$$

Solution : Quantum Error Correction. This is the science of encoding the information such that after the error has taken place, it is possible to recover the original information. In short, it means that there exists encoding and decoding functions E and D such that

$$\rho \xrightarrow{E} E(\rho) \longrightarrow \mathbf{S}[E(\rho)] \xrightarrow{D} \rho$$

A classical example of an error-correcting code is the repetition code, i.e.

$$0 \longrightarrow 000 \quad 1 \longrightarrow 111$$

such that whenever one of the three bits flips, we can still guess the original message by the rule of the majority, i.e.

$$1 \longrightarrow 111 \xrightarrow{flip} 110 \longrightarrow 1 \tag{1}$$

In quantum mechanics, it is, however, impossible to use that technique, i.e.

$$|\psi\rangle \not\longrightarrow |\psi\rangle|\psi\rangle|\psi\rangle$$

for $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ due to the no-cloning theorem which forbids the copying of an arbitrary state without disturbing the original one.
More generally, the problems we have to face with quantum mechanics are the following :

- No-cloning theorem
- Measurements which destroy the superposition
- Continuous errors on the state
- Phase errors

One way to implement error correction for a quantum state could be inspired by the classical repetition code :

$$\alpha|0\rangle + \beta|1\rangle \longrightarrow \alpha|0\rangle|0\rangle|0\rangle + \beta|1\rangle|1\rangle|1\rangle$$

In the same fashion as in the classical case, we could

Step 1- Measure all 3 qubits

Step 2- See which one differs from the others

Step 3- Flip the wrong qubit back

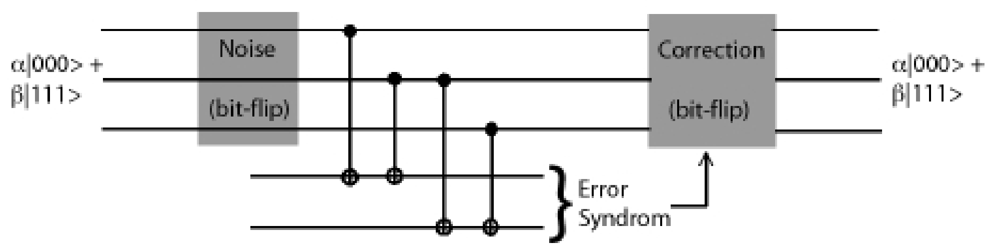
But the measurement in the first step would destroy the superposition, resulting in a quantum computation that is no more efficient than its classical counterpart. Thus, we basically want to eliminate step 1. How ?
Bit flip error
This actually corresponds, in the computational basis, to an error created by an X gate.

a) Qubit 1 and 2 are supposed to be the same - Are they?

b) Qubit 2 and 3 are supposed to be the same - Are they?

The answer to those two questions fully determines the problem (e.g., if the answer to a) is No and to b) is Yes, then we know for sure that qubit 1 differs from the two others). By the way, the set of answers ({No, Yes} in this case) is called the *Error Syndrome*.

A way to find the answer to these two questions without disturbing (i.e. reading) the state could be to use two ancillas and performing operations on these controlled by qubits 1 and 2 for the first ancilla, and 2 and 3 for the second. The operations must of course be such that it gives the answer to the above question and thus the two ancillas are actually the error syndrome. Consider the following diagram:



It is easy to verify that ancilla 1 (2) will have value 0 iff qubit 1 and 2 (2 and 3) are the same. If the differ, the value will be 1. From there, it is possible to apply bit flip operations on the three qubits controlled by the two ancillas such that the initial state is recovered.

Discrete phase flip error This corresponds, in the computational basis, to an error created by the Z gate, i.e.

$$|0\rangle \longrightarrow |0\rangle, \quad |1\rangle \longrightarrow -|1\rangle$$

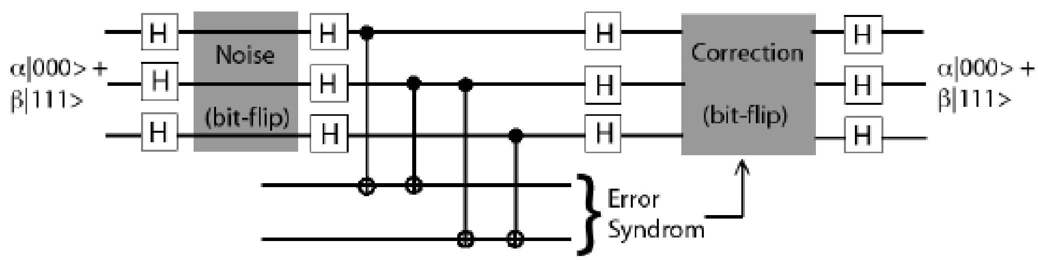
Something to notice is that

$$Z|+\rangle = |-\rangle, \quad Z|-\rangle = |+\rangle$$

where $|+\rangle = |0\rangle + |1\rangle$ (forgetting the normalization) and $|-\rangle = |0\rangle - |1\rangle$. Thus, it is clear we can use the same circuit as above, except that the encoding would be

$$|0\rangle \longrightarrow |+++\rangle, \quad |1\rangle \longrightarrow |---\rangle$$

and the circuit looks like



Now, what happens if we want to correct both flip and phase errors? We can use the 9-qubit code which is a combination of both of the above encodings, i.e.

$$\begin{aligned}
|0\rangle &\longrightarrow |\bar{0}\rangle = (|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle) \\
|1\rangle &\longrightarrow |\bar{1}\rangle = (|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)
\end{aligned}$$

It is easy to check that this encoding is able to correct an error on a single qubit of the form X, Z and Y (=iXZ), or more clearly, flip and phase error and both at the same time on the same qubit. This is a quantum error-correcting code (QECC).

Continuous phase error This type of error can be represented by

$$\begin{aligned}
R_\theta &= \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix} \\
&= e^{i\theta}\left(\cos\frac{\theta}{2}\mathbb{1} - \sin\frac{\theta}{2}Z\right) \qquad (2)
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\alpha|\bar{0}\rangle + \beta|\bar{1}\rangle &\xrightarrow{R_\theta^{(j)}} R_\theta^{(j)}(\alpha|\bar{0}\rangle + \beta|\bar{1}\rangle) \\
&= \cos\frac{\theta}{2}\mathbb{1}^{(j)}(\alpha|\bar{0}\rangle + \beta|\bar{1}\rangle) - \sin\frac{\theta}{2}Z^{(j)}(\alpha|\bar{0}\rangle + \beta|\bar{1}\rangle) \\
&\xrightarrow{EC} \cos\frac{\theta}{2}\mathbb{1}^{(j)}(\alpha|\bar{0}\rangle + \beta|\bar{1}\rangle)|\mathbb{1}\rangle - \sin\frac{\theta}{2}Z^{(j)}(\alpha|\bar{0}\rangle + \beta|\bar{1}\rangle)|Z\rangle
\end{aligned}$$

where $|\mathbb{1}\rangle$ and $|Z\rangle$ represent the error syndromes for the "identity" error or the phase error on qubit j. Thus, by doing a measurement on the error syndrome, we are left with the state

$$
\begin{aligned}
(\alpha|\bar{0}\rangle + \beta|\bar{1}\rangle)|\mathbb{1}\rangle &\quad \text{with } Prob. = |\cos\tfrac{\theta}{2}|^2 \\
Z^{(j)}(\alpha|\bar{0}\rangle + \beta|\bar{1}\rangle)|Z\rangle &\quad \text{with } Prob. = |\sin\tfrac{\theta}{2}|^2
\end{aligned}
$$

In short, it says that at the end, using the 9-qubit QECC, we either have the original state or the original state with a phase flip !!!! This development actually leads to a more general theorem :

Theorem 1. *Suppose we have a QECC* $|\psi\rangle \longrightarrow |\bar{\psi}\rangle$ *which corrects errors E and F. Then this QECC also corrects* $\alpha E + \beta F$, $\forall \alpha$, β

Proof. (Sketch) If the QECC corrects E and F, it thus means that

$$
\begin{aligned}
E|\bar{\psi}\rangle &\xrightarrow{EC} |\bar{\psi}\rangle|E\rangle \\
F|\bar{\psi}\rangle &\xrightarrow{EC} |\bar{\psi}\rangle|F\rangle
\end{aligned}
$$

where $|E\rangle$ and $|F\rangle$ are the error syndromes for errors E and F respectively. Therefore, we have

$$
\begin{aligned}
(\alpha E + \beta F)|\bar{\psi}\rangle &\xrightarrow{EC} \alpha E|\bar{\psi}\rangle|E\rangle + \beta F|\bar{\psi}\rangle|F\rangle \\
&\longrightarrow |\bar{\psi}\rangle(\alpha|E\rangle + \beta|F\rangle), \qquad (3)
\end{aligned}
$$

and thus we can recover the original state by measuring the error syndrome. □

From this, we can thus conclude that the 9-qubit code can correct *any* unitary operation on a single qubit, not just the discrete errors X, Y, and Z.