Problem 1. Price of a stabilizer code

The price $p$ of a stabilizer code is the volume of the smallest subsystem of qubits which supports all the logical operators.

(a) Find the price of the toric code defined with $N = 2L^2$ qubits.

(b) Find the price of the 15-qubit code.

(c) Prove that $p \leq n - d + 1$ and $p \geq k + d - 1$. (Hint: use the duality relation for the first inequality, and use the argument for proving the quantum Singleton bound for the second inequality).

Problem 2. Bound on local classical codes

Consider a classical stabilizer code in $D$ dimensions. Show that

$$kd^{D-1} \leq O(n).$$

Here stabilizer generators are tensor products of Pauli-$Z$ operators, and the classical code distance $d$ is the smallest subsystem of “qubits” which supports all the $X$-type logical operators.

Problem 3. Symmetry in a stabilizer code

Consider a stabilizer code with $k = 1$ defined on a $D$-dimensional hypercubic lattice ($N = L^D$). Assume that the stabilizer group $\mathcal{S}$ is invariant under finite translations:

$$T_{c_1}^1(\mathcal{S}) = \cdots = T_{c_D}^D(\mathcal{S}) = \mathcal{S}$$

where $T_j$ are operators that shift qubits in the direction of $\hat{j}$. Here $c_j$ are $O(1)$ constants. Let $d_X, d_Z$ be the sizes of the smallest subsystem of qubits which support a logical-$X$ and logical-$Z$ operators respectively. Show that

$$d_X d_Z \geq O(N).$$

(Hint: We do not need to assume locality of stabilizer generators in this problem).