Problem 1. Circuit complexity of the GHZ state

In the lecture, we showed that a constant-depth quantum circuit cannot create a ground state of the toric code from a product state \( |0\rangle^\otimes n \). Prove a similar statement for an \( n \)-qubit GHZ state

\[
|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}(|0\cdots 0\rangle + |1\cdots 1\rangle).
\]

Problem 2. Hypercube code and multi-qubit control-\( Z \) gates

This problem is a simpler version of section 4 of arXiv:1503.02065. Consider the following single-qubit phase operator:

\[
\mathcal{R}_m = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\frac{2\pi}{2m}} \end{bmatrix} = |0\rangle\langle 0| + e^{i\frac{2\pi}{2m}} |1\rangle\langle 1| \quad m \text{ is a non-negative integer.}
\]

Here \( \mathcal{R}_0 = I \), \( \mathcal{R}_1 = Z \), \( \mathcal{R}_2 = S \) and \( \mathcal{R}_3 = T \). In the lecture, we learned that the \( D \)-dimensional topological color code with certain boundaries has a single logical qubit (\( k = 1 \)) with the following transversal logical operator:

\[
\overline{\mathcal{R}}_D = (\mathcal{R}_D)^{\otimes n_{\text{odd}}} \otimes (\mathcal{R}_D)^{\otimes n_{\text{even}}}
\]

where \( n_{\text{odd}} \) and \( n_{\text{even}} \) are the number of qubits at odd and even sites when the lattice is viewed as a bipartite graph. Namely, we showed that \( \overline{\mathcal{R}}_D \) acts as a logical \( \mathcal{R}_D \) (or \( \mathcal{R}_D^\dagger \)) operator. We also learned that the smallest realization is the so-called \( D \)-th level Reed-Muller code.

In this problem, we treat the cases where the color code has multiple logical qubits. The code below is the smallest realization of the \( D \)-dimensional topological color code with \( k = D \) logical qubits. Consider a stabilizer code defined on a \( d \)-dimensional hypercube with \( n = 2^D \) qubits living on vertices. The code has only one \( X \)-type stabilizer generator, \( X^\otimes n \), acting on all the qubits, while \( Z \)-type stabilizer generators are four-body and are defined on each two-dimensional face. Two-dimensional and three-
dimensional examples are shown below:

![3D examples](image)

In three dimensions, there are six $Z$-type stabilizers. But not all of them are independent!

(The three-dimensional code has eight qubits, and has a transversal non-Clifford gate as we show below. To the best of my knowledge, this is the smallest qubit stabilizer code with such a property).

(a) Let us define a commutator of two unitary operators as follows:

$$K(V, W) = V W V^\dagger W^\dagger.$$  \hfill (5)

Show that

$$K(R_m, X) \propto R_{m-1} \quad \text{for all} \quad m \geq 1. \hfill (6)$$

(b) Consider a Hilbert space of $m$ qubits. Let $X_1$ be a Pauli-$X$ acting on the first qubit. Let us define a multi-qubit Control-$Z$ gate as follows:

$$C^\otimes m Z |j_1, \ldots, j_d\rangle = (-1)^{j_1 \cdots j_m} |j_1, \ldots, j_d\rangle \quad j_m = 0, 1. \hfill (7)$$

Here $j_1 \cdots j_m$ means a product of $j_1, \ldots, j_m$. Compute the commutator $K(C^\otimes m Z, X_1)$.

(c) Show that the code has $D$ logical qubits. Show that the code distance (minimal weight of a non-trivial logical operator) is two.

(d) Show that $R_d = (R_d)^\otimes n$ is a logical operator of the code. Also show that it acts as a logical $C^\otimes (d-1) Z$ gate. If you find this problem difficult, you can do the $D = 3$ case only.