Braneworld Cosmology Beyond the Low-energy Limit

Claudia de Rham

Girton College,

University of Cambridge



This dissertation is submitted for the degree of Doctor of Philosophy August 2005 As I was walking up the stair, I met a man who wasn't there, He wasn't there again today, I wish, I wish he'd stay away.

> As I was sitting in my chair, I knew the bottom wasn't there, Nor legs nor back, but I just sat, Ignoring little things like that.

> > Hughes Mearns

A Tosca,

Robin,

Palmitoa,

et vos Cousins,

for being there.

Declaration

This dissertation is my own work and contains nothing which is the outcome of work done in collaboration with others, except as specified in the text and Acknowledgements.

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Date: October 7, 2005 Signed:

Claudia de Rham

Fernando Quevedo

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Abstract

Advances in string/M-theory have recently motivated the study of braneworld scenarios for which our Universe would be embedded in compactified extra dimensions. Playing the rôle of a toy model, the five-dimensional Randall Sundrum scenario is of special interest. In that model, the extra dimension is compactified on an S_1/\mathbb{Z}_2 orbifold, with two three-branes on the fixed point of the \mathbb{Z}_2 symmetry.

In this thesis, we develop a four-dimensional effective theory for Randall Sundrum models which allows us to calculate long wavelength adiabatic perturbations in a regime where the ρ^2 terms characteristic of braneworld cosmology are significant. This extends previous work employing the moduli space approximation. We extend the treatment of the system to include higher derivative corrections present in the context of braneworld cosmology. The developed formalism allows us to study perturbations beyond the general long wavelength, slow-velocity regime to which the usual moduli approximation is restricted. It enables us to extend the study to a wide range of braneworld cosmology models for which the extra terms play a significant rôle. As an example we discuss high energy inflation on the brane and analyse the key observational features that distinguish braneworlds from ordinary inflation by considering scalar and tensor perturbations as well as non-gaussianities. We also compare inflation and Cyclic models and study how they can be distinguished in terms of these corrections.

We then focus on the study of the Randall Sundrum scenario in the case where the boundary branes are very close. In that regime, we obtain an effective theory, correct to all orders in brane velocity. The resulting theory is derived via recursive differentiations of the five-dimensional equations of motion. It extends the low-energy effective theory to the high energy regime. In the case of cosmological symmetry the theory reproduces the fivedimensional behaviour exactly, which the low-energy theory fails to do. This extension has the remarkable property of including corrections only as powers of first order derivatives. This important feature makes the theory particularly easy to solve. Perturbations in the tensor and scalar sectors are then studied. When the branes are moving, the effective Newtonian constant on the brane is shown to depend both on the separation of the branes and on their velocity. In the small distance limit, we compute the exact dependence between the four-dimensional and the five-dimensional Newtonian constants. We then extend the theory by introducing a potential for the radion and show that for a fixed background, the perturbations propagate the same way as they would in a standard four-dimensional scenario.

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Chapter 1

Introduction



Arbol, the Shape of the Universe by the Mayas

1.1 Early Universe Cosmology

1.1.1 Cosmology

Throughout history, cosmology has progressively evolved from addressing questions about our environment and what surrounds us, to a scientific study of our Universe, its structure, its evolution and most intriguing of all, its origin.

The earliest view of our world was anthropomorphic, where all forces of natures were alive, then 5,000 to 20,000 years ago, myths took over to address more complicated issues of our Universe. The creation of the Universe is, for instance, addressed in Egyptian cosmology where the sky Goddess Nut feeds from Ra, (the sun God) and gives birth to Ra nine months later. Ra is thus a self-creating God, and the Universe is cyclic and eternal. The idea that our Universe might be cyclic avoids the disturbing notion of a real beginning and end of our Universe, what is before the birth of the Universe is the Universe itself. This idea was shared by the Mayas who believed in successive cycles of creation and destruction of about 5,000 years. In Mayan cosmology, the Universe is surrounded by a crocodile eating its own tail. The sky is supported by the Mayan Ceiba tree which communicates with the Underworld, or other world. As we shall see, some of these notions are still very much alive.

One of the first systematic studies of our Universe dates to the Babylonians, although no real model was developed to explain their observations. The Greeks, on the other hand, used many of their observations to create a geometrical model of our Universe, strongly based on mathematical logic, which is still the language in which most of cosmology is studied nowadays.

Although these different myths often mixed observational facts with supernatural beliefs, they all had for common aim to give a rational explanation of the everyday world, as it was observed, or as it was believed to be. Their proper logic might have been different from ours, but they were all seen as consistent stories. Even if the specific terminology and the details of the stories are not the same, many notions used nowadays in cosmology are not original to our time but have been shared by many cultures before us. In any culture and epoch, the questions puzzling us are often very similar and remain open until a "consistent" and "reassuring" model is proposed. What is consistent probably depends on which logic is used, but what is reassuring often depends to what extent one finds ones own place in the model.

Over the past 20 years, some specific questions have puzzled cosmologists and have driven them to come up with a model of the early Universe in which *inflation* plays a crucial rôle. Taking this modern approach, we shall describe which questions are at the core of "modern" cosmology and how these issues may be addressed.

1.1.2 Early Universe and Modern Puzzles

Present observations of distant supernovae strongly suggest that the present Universe is expanding and is well described by a spatially flat Friedmann Robertson Walker (FRW) metric with scale factor a. The expansion of the Universe is mainly "fed" by dark matter and a cosmological constant, as described by the Friedmann equation which relates the Universe expansion to its matter content [1]:

$$H^2 = \frac{\kappa_4}{3}\rho,\tag{1.1}$$

where ρ is the energy density for the dark matter and the cosmological constant, and the Hubble parameter $H = \dot{a}/a^{-1}$ describes the evolution of the scale factor a. If we assume that the Universe has followed the same behaviour in the past, its origin must have been singular, originating from a point around 15 billion years ago. Just after its creation, the temperature of the Universe must have been very important, (going as $T \sim a^{-1}$), leading to the Hot Big Bang model. This model is believed to give a relatively accurate description of the Universe since the time of nucleosynthesis and relies on the assumption that before this period, the Universe was composed of a hot gas [2].

Flatness Puzzle

The present Universe appears to be almost flat. In a FRW Universe, as described by eq.(1.1), any deviation from flatness would increase as the Universe expands. In other words, in order to observe such a flat Universe today, the deviation from flatness just after the Big Bang must have been tiny. The average density ρ of our universe must be very close to the critical density ρ_c which would be necessary in (1.1) to observe a perfectly flat Universe. If just after the Big Bang, $\Omega = \rho/\rho_c$ had a value slightly greater than one, its current value, 15 billion years later, would be huge and conversely if Ω had started slightly smaller than one. For Ω to be so close to one today (a value of $|\Omega - 1| \sim 0.1$ is observed),

¹In this thesis a dot designates the derivative with respect to the proper time and κ_4 represents the four-dimensional Newtonian constant.

its departure from that value just after the Big Bang must have been of order $\sim 10^{-59}$ or smaller which seems to be of a highly unlikely precision.

Magnetic Monopole Puzzle

This puzzle has at its origin the fact that no monopoles have to date been observed. Yet, they seem to be predicted in large numbers by the conventional Hot Big Bang scenario combined with Grand Unified Theories, and should dominate the energy density of our Universe before Helium synthesis [3]. Indeed, in theories that unify all forces of nature, such as Grand Unified Theory (GUT, without gravity) or superstring theory (with gravity), topological defects (such as monopoles or cosmic strings) are expected to be generated during phase transitions. At very large temperature, such as the ones in the Hot Big Bang scenario, the probability for the production of such topological defects is large, much larger than that compatible with observations. In three spatial dimensions, two cosmic strings may interact and annihilate each other. This is however not the case for monopoles. In the conventional Hot Big Bang model, monopoles are therefore expected to be found in large numbers.

Horizon Puzzle and Origin of the Large Scale Structure

The observation that physical properties, such as temperature and in particular the Cosmic Microwave Background² (CMB), appear remarkably smooth in all directions is at the origin of the horizon puzzle, (or what is called the *homogeneity* or *causality* problem) [4]. Deviations from the 2.728° K temperature of the CMB are indeed of order of 5.10^{-3} % [5]. Widely separated regions of space therefore share some of the same physical properties. Although this does not seem to be a problem today, as these regions are causally connected, when looking at these regions at the time when the radiation responsible for the CMB was emitted, they appear to be causally disconnected (as it can be more clearly seen on fig.

²Although the existence of the CMB has been predicted independently by G. Gamow in 1948, and by R. Alpher and R. Herman in 1950, it is only in 1965 that it was first observed, by accident, by A. Penzias and R. Wilson.



Figure 1.1: Horizon problem: regions with the same temperature were causally disconnected at the time where the radiation was emitted.

1.1). If the Universe was around one million years old at the time of emission, the distance between these regions would have been about 100 million light years *ie.* much greater than the Hubble length scale, making it impossible for these regions to interact and therefore to explain why their temperature could have been so close.

Although the CMB appears to be homogeneous with fluctuations of order 10^{-3} %, surprising results may be observed from the CMB data once the uniform value of 2.728° K is subtracted from all regions. In particular the study of the remaining anisotropies, presents a remarkable large scale structure [6]. A possible explanation for the origin of this large scale structure was first given using cosmic strings. But although the perturbations generated by cosmic strings are almost scale invariant, their exact spectrum is not compatible with observations. The origin of the large scale structure is thus not well defined in the standard Hot Big Bang model, even if some potential candidates have been proposed.

Origin of the Big Bang

In the Hot Big Bang model, the origin of the Universe is singular. Although the different phases of the Universe can be understood with good accuracy, and its evolution can be followed between 1s after the singularity until the present time, no explanation is given on the origin of the Big Bang itself. In the Hot Big Bang scenario, the origin of the Universe is therefore only postponed by the presence of the singularity, what happens before that and what generates it remains mysterious. It is actually extremely difficult to make much progress on that subject, although this is one of the most intriguing questions. So far very few acceptable ideas have yet been proposed. As an example, we may suggest the idea that our Universe might have been created by quantum tunnelling from nothing [7]. This proposal relies on the existence of a Hawking-Moss instanton from which the Universe could have been self-created.

Dimensionality of our Universe

So far, no remarks have been made concerning the dimensionality of our Universe. From the every day life point of view, no objections can be made to the fact that we appear to live in a (3 + 1)-dimensional spacetime, and to date no deviation from four-dimensional physics have yet been observed, not even at very small physical scales. Since the concept of extra-dimensions is very hard to picture, the fact that only three spatial dimensions are observed did not seem to present any problem before models requiring more spatial dimensions have been suggested. The Hot Big Bang scenario does not give any explanation for this fact and, as we shall see, very few explanations have actually been proposed. An anthropic argument might answer half the question: If our spacetime had less than three spatial dimensions, it is reasonable to assume that life would probably not have developed and the question of the origin and dimensionality of that Universe would not be posed. For superstring theory to be a consistent theory, extra-dimensions have to be considered, as we shall see. The reason why only three of these dimensions are observed remains uncertain but a more detailed discussion on this subject and on some proposed mechanisms is given in section 1.3.2.

As we shall see in what follows, an explanation for some of these puzzles is given in the model of inflation which will be described now. This is no proof of the validity of inflation nor of the uniqueness of such a model but it is however reassuring to find a model capable of explaining some of the main features of cosmology, although many questions remain open and some new puzzles arise.

1.1.3 Inflation

de Sitter Cosmology

The idea behind inflation relies on the possibility that our Universe underwent a nearly de Sitter phase prior to the usual Hot Big Bang evolution. Inflation is not a substitute of the Hot Big Bang scenario but rather a mechanism that sets the initial conditions for the usual evolution. Such a phase would be generated if the right hand side of the Friedmann equation (1.1) was a constant, the scale factor would then grow exponentially, leading to a de Sitter cosmology [8]. This is for instance the case if the only energy density present (or the dominant one), arises from a uniform vacuum energy Λ . The scale factor would then grow in proper time as $a(t) \sim e^{Ht}$ where the Hubble parameter H is constant: $H = \sqrt{\kappa_4 \Lambda/3}$, leading to a uniform expansion of the Universe. Such an evolution has the advantage of diluting any deviations from flatness as well as any topological defects, giving an explanation for two of the previous puzzles, although inflation was first proposed only to solve the monopole problem. Its capacity to solve some of the other puzzles has made this model specially interesting.

If we suppose that some curvature k was present at the beginning of the Universe, this would give rise to the following Friedmann equation:

$$H^2 = \frac{\kappa_4}{3}\Lambda - \frac{k}{a^2}.$$
(1.2)

The effective energy density associated with this curvature, $\rho_k = -3k/\kappa_4 a^2$, would be exponentially suppressed in comparison with the energy density for the vacuum: $\rho_k/\rho_\Lambda \sim a^{-2} \sim e^{-2Ht}$. Any deviations in flatness are therefore exponentially suppressed by the exponential expansion of the scale factor, explaining why the present Universe is observed to be so flat. Similarly, in such a scenario, topological defects such as magnetic monopoles could still be produced, but their energy density would be exponentially suppressed in a de Sitter spacetime, explaining why so few (or none) have yet been observed.

Although the scale factor may be very small in a de Sitter spacetime, the time T needed

for it to go to zero back in time is infinite:

$$T = \int_0^{a_0} \frac{\mathrm{d}a}{\dot{a}} = H^{-1} \int_0^{a_0} \frac{\mathrm{d}a}{a} = \infty, \tag{1.3}$$

in a purely de Sitter spacetime, it would thus take an infinite proper time to go back to the singular pointlike Big Bang origin. Since the geometry is actually only approximatively de Sitter, the actual time to go back to the Big Bang is very large, but *finite*. This would still be enough to solve the horizon problem, if at some point before the usual Hot Big Bang evolution, the Hubble scale had been greater than the physical scale. This would be possible if $\partial_t (aH)^{-1} < 0$, *ie.* if the comoving Hubble length was decreasing in time. This is in general one of the conditions required for inflation. The Hubble scale can hence be greater than the physical scale at the beginning of inflation, and so interactions within a region of the size of the physical scale would be possible. This would hence explain the large scale homogeneity of the CMB.

A possible explanation for the flatness, the horizon and the monopole puzzles would therefore be available if our Universe underwent an inflationary period of about 60 efolds after the Big Bang [8]. Since this is one of the most commonly accepted explanations, we shall describe below how such a phase could arise.

Inflaton Scalar Field

In order to obtain a nearly de Sitter phase, one needs a vacuum energy density which dominates the content of the Universe. For a conserved fluid to have such properties, its pressure $p = \omega \rho$ should be related to its energy density ρ in such a way that ρ does not vary. From the conservation of energy, $\dot{\rho} = -3H(\rho + p)$, this fluid needs to have a pressure $p \simeq -\rho$, in other words, the parameter w for the equation of state should be very close to $w \simeq -1$. This is possible if a scalar field φ is present in the theory and dominates the contributions to ρ and p. The stress-energy tensor will then derive from the action:

$$S_{\varphi} = \int \mathrm{d}x^4 \sqrt{-q} \left(\frac{1}{2} \left(\partial \varphi \right)^2 + V(\varphi) \right), \qquad (1.4)$$

where the potential $V(\varphi)$ describes the scalar field interactions [9, 10]. Assuming cosmological symmetry, the energy density for this scalar field is $\rho = \frac{1}{2}\dot{\varphi}^2 + V(\varphi)$ whereas its pressure is $p = \frac{1}{2}\dot{\varphi}^2 - V(\varphi)$. In the regime where the scalar field is evolving very slowly, its kinetic energy may be neglected in comparison with the potential energy, $\dot{\varphi}^2 \ll V(\varphi)$, and the required property $p \simeq -\rho$ is thus satisfied. This leads to an almost constant Hubble parameter $H^2 \simeq \frac{\kappa_4}{3}V(\varphi_0)$, as long as the field is moving slowly $\varphi \simeq \varphi_0$. This scalar field is often called the inflaton.

In this model, the contribution of the potential should be very important, so the scalar field can not be stationary at a minimum located at V = 0, (which would be for instance the minimum for a free massive scalar field with potential $V = \frac{1}{2}m^2\varphi^2$). Instead one can imagine a scenario for which the scalar field is located at a non-vanishing minimum of the potential. Including temperature corrections, the effective potential may spontaneously change configuration and new vacua can appear. The scalar field can hence tunnel to a new vacuum after having stayed (for about 60 efolds) in the false vacuum with non-vanishing vacuum value. In that model, the assumption that the scalar field moves slowly (or not at all) would hence be valid but we shall see later that this idea does not work. In many scenarios, the scalar field is instead assumed to star "up the hill" of a slowly-varying potential and rolls slowly towards the minimum, although few explanations on why the scalar field would start up the hill are given. The potential should then be very flat and satisfy some slow-roll constraints [11].

Predictions of Inflation

Although inflation seems to solve three of the previous puzzles, some questions remain open. The origin of the Big Bang, for instance, is still unaddressed. Inflation has however the remarkable property to solve a supplementary question which is the origin of the large scale structure and to model with a surprising accuracy some of the large scale structure details.

In the inflationary scenario, the large scale structure finds its origin in the curvature

perturbations (sourced by the scalar field perturbations). The inflaton is indeed expected to have very small thermal fluctuations which should be almost the same at every scale and have a gaussian distribution. The subsequent fluctuations in the curvature could then explain the anisotropies observed in the CMB measurements [12,13]. The power spectrum of such perturbations generated within an accelerating de Sitter background is surprisingly close to current measurements [14,15]. Although one might argue that other theories might give the same prediction [16, 17], this represents a very important prediction of inflation and is the reason why it has been the subject of so much interest despite the presence of some problems (which we shall summarise in the following).

So far no uncontested alternative theories have been proposed and although the original inflation scenario has found some failures, some more complicated versions have been proposed. Among these models, we may in particular point out *hybrid inflation*, which is generated by two scalar fields, one responsible for the large vacuum value and the other one for the thermal perturbations [18]. After tuning, these more elaborated versions of inflation could model a wide range of different large scale structure observations.

Problems of Inflation

• Interpretation of the inflaton scalar field

One of the most fundamental problems of inflation is the origin of the inflaton scalar field as no such fundamental spin-0 particle has yet been observed. In the original scenario [19], A. Guth suggested that the inflaton scalar field could be identify as one of the fields present on the SU(5) GUT as the Higgs field [20]. In this idea, the non-vanishing vacuum value for the inflaton would be obtained through the false vacuum mechanism we explained earlier on. However in that model, the amplitude of the false vacuum should be tuned to a value much beyond what would be natural from the GUT scales. Furthermore the decay rate of the false vacuum would be very important. The interpretation of the inflaton as a possible Higgs candidate has therefore been dropped. Some other attempts have been given but so far the question remains opens.

• Fine-Tuning

Another fundamental problem of inflation is its requirement of impressive fine-tuning in its different parameters and initial conditions. The potential needs indeed to satisfy some slow-roll conditions and, at the same time, should be capable of generating 60 efolds of inflation. The original value of the potential should thus be enormous and especially compared to the present cosmological constant. It seems highly improbable that such fine-tuned quantities just happened to take place and to provide the Universe with such convenient features. The physical intuition seems to suggest that some fundamental underlying physics is not understood and should explain why such value could have been picked or provide an alternative mechanism.

• Inflation in question

The previous two problems have put inflation into question despite its accurate predictions of the large scale structure. In particular the way the initial conditions should be imposed is subject to important polemics especially since the origin of inflation itself is not fully understood.

We have mentioned so far only the two main problems of inflation but some other criticisms exist. In particular, inflation does dilute the topological defects but once inflation has ended, monopoles could still come back in large numbers. Another open question is how does inflation end? In the original idea of A. Guth, a natural ending would occur when the scalar field spontaneously decays into the true vacuum, however this idea is no longer considered. Instead the scalar field is often assumed to roll gently down the potential hill and it is difficult to find a natural ending to inflation. In models of hybrid inflation, such an end is more easy to track since one scalar field is directly responsible for the shape of the potential.

Despite the success of inflation to explain some of the most important puzzles of cosmology, some questions remain open. Since the idea of inflation was first suggested, 24 years ago, enormous efforts have been placed into this area to extend this model, fill the open questions and test its predictions by important observation programs. Despite all these efforts no real alternative or explanation can yet be given although many new important ideas are in the process of being tested. String theory, born more than 30 years ago and followed by Superstring theory 25 years ago has gained an incredible interest as a fundamental unified theory. The idea that gravity could be associated with other forces (for instance electromagnetism) in a unified theory is however not new and dates from Kaluza Klein compactification. As we shall see, these unified models require extra-dimensions [21] whose implications for cosmology shall be the main focus of this thesis (for a review on extradimensions, see [22]).

1.2 Extra-dimensions and Braneworld

1.2.1 Kaluza Klein Compactification and Extra Dimensions

The notion that our Universe could be embedded in compactified extra-dimensions is not new. In 1919, T. Kaluza suggested that our Universe could have actually been fivedimensional with the fourth spatial dimension curled into a circle of radius $R \ll H^{-1}$ where H^{-1} is the horizon scale [23]. If R was much larger than our horizon scale, our Universe would effectively be five-dimensional. When $R \sim H^{-1}$, we would not see the fifth dimension exactly as one of our three spatial dimensions, but some effects would be present. In particular, to go from one point to another, light could circle on a geodesic around the compactified dimension any number of times. It would therefore seem to us, that any object would be repeated with a certain periodicity depending on R. In the case where the radius of the extra-dimension is of order the Planck scale, we expect to recover an effectively four-dimensional spacetime as we observe. In that case, the energy required to excite modes in the extra-dimension is huge and, at low energies, no such "Kaluza Klein" modes are excited.



Figure 1.2: Kaluza Klein compactification.

In Kaluza theory, the five-dimensional spacetime would have the metric:

$$ds^{2} = \Phi dy^{2} + 2A_{\mu} dy dx^{\mu} + g_{\mu\nu} dx^{\mu} dx^{\nu}, \qquad (1.5)$$

where y designates the extra-dimension and x^{μ} our four-dimensional spacetime. Only gravity can propagate in the fifth dimension but on our four-dimensional spacetime, electromagnetism is recovered as a consequence of the presence of the vector field A_{μ} . A_{μ} could indeed play the rôle of the photon, Φ is what is called the dilaton scalar field, and $g_{\mu\nu}$ plays the rôle of the graviton. From five-dimensional general relativity, we recover Einstein gravity and Maxwell electromagnetism. An extension of this model was performed in 1926 by O. Klein [24]. In that theory, a pointlike particle in four dimensions is described as a circle wrapping the fifth dimension. Excitations along this circle (or string) can be responsible for different characteristics of the pointlike particle. This idea is at the basis of string theory. In string theory, the higher-dimensionality of spacetime is not a possibility but a requirement. We shall see for instance in superstring theory, that our Universe should be ten-dimensional. Since only three spatial dimensions are effectively observed, one might think that the extra-dimensions should be compactified. However there exists an alternative: The notion of branes, that we shall describe later. The notion of a domain wall in an extra-dimension can be understood independently to string theory, however the existence of branes has been motivated by string theory where they represent a hypersurface on which open strings end. We shall therefore very briefly review the basic notions of string theory in the next subsection and see how the notion of branes has emerged.

1.2.2 Basic notions of String Theory

Conformal Invariance and Critical Dimension

At the basis of string theory, is the notion we have already mentioned that particles might not be pointlike but might be extended objects with their own dimension [25–28]. A particle line element is given by $\sqrt{-G_{ab}\dot{X}^a\dot{X}^b}$ (where a dot is a derivative with respect to its proper time), X^a being the coordinate of the particle in the (d + 1)-dimensional target space and G_{ab} its metric. By analogy, the worldvolume of an extended object is instead the square root of the determinant of the tensor $V_{\alpha\beta} = -G_{ab} \partial_{\sigma^{\alpha}} X^a \partial_{\sigma^{\beta}} X^b$ where $\sigma^0 = \tau$ is the proper time of the object and the other σ^I represent its proper spatial dimensions. In particular, a string has only one spatial dimension σ^1 and the tensor $V_{\alpha\beta}$ is two-dimensional. A priori, there is no reason to consider a string rather than a membrane (with two spatial proper dimensions) or any other extended objects, but we shall see that strings have the very remarkable property to preserve conformal invariance, which is not the case for any other extended object. Furthermore, despite attempts which have lead to Matrix theory [29], it has not yet been possible to give a consistent quantised theory for extended objects with dimensionality higher than two. Finally, string theory is the only theory able to unify gravity with the other forces of nature. This is the reason why strings play a very special rôle in modern physics.

The starting point for the study of string theory is the Nambu-Goto action [30] which, by analogy to a pointlike particle's action, is given by

$$S = -T \int d^2\sigma \left[\det \left(-G_{ab} \,\partial_{\sigma^{\alpha}} X^a \,\partial_{\sigma^{\beta}} X^b \right)^{1/2} \right], \tag{1.6}$$

where $T = 2\pi \alpha'$ is the string tension and α' , which has the dimensionality of a length

square, is the fundamental parameter of the theory. Although this action is more intuitive to derive by analogy to the point-particle action, the presence of the square root makes it hard to quantise. Instead we may derive the same equations of motion from the Polyakov action [26]:

$$S = -\frac{T}{2} \int d^2 \sigma \sqrt{-h} G_{ab} h^{\alpha\beta} \partial_{\sigma^{\alpha}} X^a \partial_{\sigma^{\beta}} X^b, \qquad (1.7)$$

where the worldsheet metric $h_{\alpha\beta}$ appears as a Lagrange multiplier in this new action imposing the constraint $X_{,\alpha} \cdot X_{,\beta} - \frac{1}{2} h^{\gamma \delta} X_{,\gamma} \cdot X_{,\delta} h_{\alpha\beta} = 0$, (where contractions are performed with respect to G_{ab}). This implies the conformal relation between the worldsheet metric and the induced metric: $h_{\alpha\beta} = \Omega^2 X_{,\alpha} \cdot X_{,\beta}$, with an arbitrary conformal factor Ω . This new Polyakov action is clearly invariant under the conformal transformation $h_{\alpha\beta} \to \Omega^2 h_{\alpha\beta}$, which would not be the case if the worldsheet was not two-dimensional. This theory is therefore conformally invariant and is a special case of a conformally invariant sigma model [31]. When quantizing this theory, it seems therefore natural to require that this symmetry remains preserved. Although this is trivial at the classical level, at the quantum level, some quantum anomalies break that symmetry [25, 26, 31]. (Quantum anomalies are generic to quantum field theories. They arise from regularisation which have to be made at the quantum level in order to calculate correlation functions, since operators can not be evaluated at the same point.) For bosonic string theory, it is however possible to cancel the quantum anomalies and to preserve the conformal symmetry if the dimension of the target spacetime is fixed to 26 [32,33]. This number seems to come out of nowhere, and in its derivation, it appears to be an integer only by chance, however this is a very important result and we shall assume in what follows, that we live in a target spacetime with the critical number of dimensions for the theory to be conformally invariant at the quantum level.

Boundary Conditions

In the actions (1.6) or (1.7), some boundary conditions need to be specified for the string coordinates X^a . In particular we might either think of periodic conditions or Neumann

boundary conditions which are both consistent with the variation principle. When the strings are quantised, the different excitation levels lead to different particles.

- For Neumann boundary conditions, the string is open, and the condition $\partial_{\sigma} X^a = 0$ is imposed at its end points. In that case, the mass square of the string is given by $M^2 = (-1 + n) / \alpha'$ where n is the excitation number of the string. In particular, when the open string is in its vacuum state (no excitation), its mass square is actually negative, leading to the presence of a tachyon. It is the presence of this tachyon which makes bosonic string theory ill-defined. For n = 1, the string is massless. This degree of excitation may be parametrised by a vector A_a which defines along which direction in the target space the string is excited. From the target space point of view, this corresponds therefore to a massless vector field, in other words this string could be interpreted as a photon.
- For Periodic boundary conditions, the string is closed and $X^a(\sigma, \tau) = X^a(\sigma + 2\pi, \tau)$, if we assume its total length to be 2π . For such a string, excitations can propagate in two possible directions (clockwise or anticlockwise, which are called left or right movers), without interfering with each other. The number of right and left excitations need however to be the same, even if the directions of excitation on the target space are not necessarily the same. The mass square of the string is then given by $M^2 =$ $\left(-1 + \frac{1}{2}(n_L + n_R)\right)/\alpha'$ where $n_{L,R}$ are the excitation numbers on both directions, $n_L = n_R$. The first excited state of a closed string is therefore massless and is parametrised by a tensor field (with one index for each right and left moving mode). This tensor field may be decomposed as a traceless and symmetric tensor S_{ab} , an antisymmetric one B_{ab} and a scalar field S'. S_{ab} represents a spin-2 massless particle and is hence a good candidate for the graviton. String theory (or superstring theory) is the only theory which incorporates gravity. The antisymmetric tensor field B_{ab} represents what is called the Kalb-Ramond field and S' the dilaton scalar field, both these fields are massless.

Although this is only the basic description of bosonic string theory, (which presents various problems), the same general ideas remain valid when its extension to superstring theory is considered. In particular, the presence of the graviton is a key element of string theory. The notion of a curved spacetime can for instance be interpreted as a flat Minkowski spacetime on which closed graviton strings are propagating.

Superstring Theories

Despite being the first theory capable of incorporating gravity in a natural way [34], being unstable (due to the presence of the tachyon scalar field), and lacking of any formalism to describe fermions, the bosonic string is not of great interest. Its extension to superstring models [35] and to heterotic strings, on the other hand, has gained a very popular interest. Today five different superstring theories have been found: the Type I, with $\mathcal{N} = 1$ supersymmetry followed by the Type IIA and IIB theories with $\mathcal{N} = 2$ supersymmetry and the two Heterotic SO(32) and $E8 \times E8$ string theories. For these theories, the critical target space dimension is 10 rather than 26 [36]. In 1984, M. Green and J. Schwarz indeed showed that in ten dimensions, not only the anomalies related with the conformal symmetry of the worldsheet cancel, but as well the target space is free of chiral or gravitational anomalies for some special groups [37].

The main difference between these theories relies on which boundary conditions are imposed (fermions have more complicated boundary conditions), and whether the strings are oriented or not. In heterotic strings, which are theories for closed strings excursively, the left and right movers do not belong to the same theory, one satisfies superstring conditions, while the other belongs to the bosonic sector. The heterotic theories however remain tendimensional. Among these theories the only one that contains open strings is the Type I, all the others are exclusive to closed strings. The open string need however to contain closed strings as well since an open string may close itself up and form a closed string. Due to the fermionic nature of superstrings, open strings have charges (or Chan-Paton factors) attached at their end points.

T-duality

The last crucial point we shall briefly mention about string theory is the notion of Tduality [38]. We have seen that particles may be interpreted as strings with different vibration or excitation numbers. Furthermore for the theory to make sense, the dimension of the target spacetime should be 10 (or 26)-dimensional. Since we only observe three spatial dimensions, it makes sense to consider that some of these dimensions might be compactified on a circle of radius R, just as the Kaluza Klein compactification. Strings might therefore wind around these compactified dimensions. The energy needed for a particle to wind around a compactified dimension is proportional to R. Inversely, for a particle to be excited along the compactified dimension, the energy required goes as α'/R . If we consider a string with m winding modes, and n excitation number along that compactified direction, its energy will therefore be the same as the one of a string with nwinding modes around a compactified direction of radius α'/R and excitation number m. Torii of radius R will therefore share the same energy spectrum as torii of radius α'/R , with the only difference that the rôle of winding modes and vibrations modes will be inverted. The duality $R \rightarrow \alpha'/R$ plays a crucial rôle in string theory and in particular the fixed point of the symmetry, at $R = \sqrt{\alpha'}$, suggests the existence of a smallest length scale for theories of closed strings on a torus (since any length scale smaller than $\sqrt{\alpha'}$ is equivalent to a length scale bigger than $\sqrt{\alpha'}$, although some branes could probe smaller scales in other situations.

This T-duality actually relates the different closed superstring theories:

type IIA $\stackrel{\text{T-duality}}{\longleftrightarrow}$ type IIB heterotic $SO(32) \iff$ heterotic $E8 \times E8$.

1.2.3 M-theory and Orbifold Branes

As already mentioned, five well-defined different string theories exist. We might therefore ask the question whether these theories are really independent or if they are not part of the same fundamental theory and represent different limit of it. This is a sensible question since T-duality relates some of them. Furthermore only a perturbative approach to string theory is known. The underlying fundamental theory could then be the answer of the quest of a non-perturbative approach to string theory incorporating the five different string theories known so far. This theory has been named "*M-theory*", but apart from its known string theory limits, very little is actually known about it, (for a review, see [39]). It has been shown that an additional limit of M-theory could be found, and at low-energy, M-theory should look like 11-dimensional supergravity, (see for instance [40]). The underlying fundamental theory should therefore be 11-dimensional, and appear ten-dimensional only in some special limit such as for the five different string theories [41].

Any object in the ten-dimensional string theory description should therefore behave as an extended object in the 11-dimensional M-theory. Thus, a string in the ten-dimensional string theory should, for instance, look like a (2 + 1)-dimensional membrane in M-theory, and similarly for any D*p*-brane in string theory. In particular, a special interest can be given to what the heterotic $E8 \times E8$ should look like from the 11-dimensional M-theory point of view. In 1996, P. Hořava and E. Witten suggested that from the extended Mtheory point of view, the heterotic $E8 \times E8$ string theory would appear as a Heterotic M-theory, compactified on a S^1/\mathbb{Z}_2 orbifold with orbifold-branes located at the fixed point of the symmetry [42]. The extra-dimensional should be larger than the other compactified dimensions (compactified on a Calabi-Yau), suggesting that there could be a regime where the Universe would appear five-dimensional. This model has opened an entire new set of possibilities for the study of orbifold branes. It is motivated by advances in this subject, that so much interest has been given to braneworld cosmology. The main part of this thesis is indeed devoted to the study of orbifold branes in the Randall Sundrum model which is a very simple five-dimensional version of the real Heterotic M-theory model.

It might seem strange that the same fundamental theory would be capable of describing all of the five superstring theories, especially when one of them contains open strings while the others not. The reason for that is the existence of another duality, the weak/strong coupling duality (or S-duality), that relates the different theories [43]. In particular, the heterotic SO(32) theory is dual to the open (and closed) type I superstring theory. At weak coupling, open strings in the heterotic sector might exist but they are highly unstable and hence quickly decay into closed strings. The type I theory might be the strong coupling limit of some region of M-theory. In order to incorporate open strings in a more fundamental way to the theory, we shall see that D-branes should be considered.

1.2.4 D-Branes as the End Points for Open Strings

Dirichlet Boundary Conditions

S-duality relates the Type I open string theory to the heterotic SO(32) closed string theory. The notion of T-duality on open strings is however less clear. For T-duality to work, the string needs to wind around a compactified direction and not be able to unwind without interactions. When Neumann boundary conditions are imposed at the end point of strings (requiring their spatial momentum $\partial_{\sigma} X^a$ to vanish at that point), open strings can unwind without any interaction. However this would not be the case if their end points were instead fixed. For that, Dirichlet boundary conditions may be imposed, and T-duality could have for effect to swap Neumann and Dirichlet boundary conditions. If d is the spatial dimension of the theory (*ie.* d = 9 for superstrings), we may consider that (p + 1)of the string coordinates satisfy Neumann boundary conditions $(0 \le p \le d)$, while the end points of the remaining (d - p) coordinates are constrained to Dirichlet conditions:

$$\partial_{\sigma} X^{a}(\tau, \sigma)|_{\sigma=0,\pi} = 0, \qquad a = 0, \cdots, p \qquad (1.8)$$

$$X^{b}(\tau,\sigma)|_{\sigma=0,\pi} = \bar{X}^{b}(\tau), \quad b=p+1,\cdots,d.$$
 (1.9)

The end point of the string is therefore fixed on a p-dimensional hypersurface called D(irichlet)p-brane [28,44]. The study of D-branes will present interesting links with gauge field theories. In the previous example (1.8 - 1.9), the boundary conditions were taken to be the same on both ends of the string, but there is actually no reason to impose this condition and we might as well take into consideration "mixed" coordinates for which a Neumann condition is imposed at one end and a Dirichlet one at the other end. The special

case considered until now, for which all coordinates had Neumann boundary conditions, represents the situation of spacetime entirely filled with a Dd-brane. In that case the strings are completely free to move in the entire spacetime.



Figure 1.3: Closed strings propagate in the entire spacetime while the end points of open strings are attached to *D*-branes.

Maxwell Gauge Theory

After quantising the open strings on D*p*-branes, we may consider the first excited state (analogous to the photon massless state we presented earlier). If the string is excited along the directions tangent to the D*p*-brane, a (p + 1)-dimensional massless vector field A_a will parametrise the vibrations. This field will have (p - 1) independent variables. From the D*p*-brane perspective, this field will therefore play the rôle of a Maxwell gauge field. If on the other hand, the string was excited along the direction normal to the D*p*-brane, the (d - p)-dimensional vector field A_b , parameterising the vibrations, will not carry any
indices along the brane. This field having (d-p) independent variables, the D*p*-brane will therefore see (d-p) massless scalar fields. We therefore recover in a completely natural way a *p*-dimensional Maxwell gauge theory on the D*p*-brane. This is a remarkable result, explaining the special interest which has been given to these objects.

So far we have only considered one Dp-brane alone. If we extended this model to N parallel D-branes, we could see that the same result would remain valid, and in particular if N such D-branes were coincident, they will carry a U(N) gauge field.

D-branes are hence extended objects of superstring theory which carry charges (electric and magnetic Ramond-Ramond charges) [45] and present very interesting properties. In order to understand their implications to cosmology some special cases should be considered, and the model will be largely simplified.

1.2.5 Hierarchy Problem and Large Extra-dimensions

Hierarchy Problem

One of the most confusing problem of particle physics is why two fundamental quantities, sharing the same unit, do not share the same order of magnitude. Naïvely, one would expect all numbers coming from a fundamental theory to have the same order of magnitude. This is not the case for the Planck and the electroweak scales. The electroweak symmetry breaks at an energy scale of order 250 GeV (around which the vacuum expectation value of the Higgs field is hence taken). The Planck scale (at which quantum gravity effects come in), on the other hand, is of order 10^{18} GeV. The Higgs mass is hence about 10^{16} times lighter than the Planck mass.

Compactified Extra-dimensions

So far, even though we considered the spacetime to have extra dimensions, we implicitly assumed the extra dimensions to be compactified on torus of small radius à la Kaluza Klein (or compactified on a Calabi-Yau manifold). These compactifications were made so that effectively our Universe would remain four-dimensional, the radius of the extradimension being too small for Kaluza Klein modes to be excited at low energies. In these models, the four-dimensional physics is unaffected by the presence of extra-dimensions, and in particular problems occurring in usual physics (such as the hierarchy problem) remain unsolved. Although it is convenient to recover usual four-dimensional theory in some limit, the presence of these extra-dimensions may appear as "wasted".

Large Extra-dimensions

Motivated by heterotic M-theory, (where the 11th-dimension could be compactified on an S^1/\mathbb{Z}_2 -orbifold), there has recently been an increased interest to models where some extra-dimensions could be "large". If all forces of nature, but gravity, were confined on a four-dimensional hypersurface, the usual physics will apply for them. Gravity (being propagated by closed strings), on the other hand, could be allowed to evolve in the entire spacetime. This may be justified by the fact that gravity is the weakest force of nature at short-distances and has been tested only up to distances of order of a millimetre³ [47]. It is hence possible that departures from Newton's law might occur at lower scales, which would be the case if gravity was not confined to a four-dimensional spacetime. This new idea, suggested by N. Arkani-Hamed, S. Dimopoulos and G. Dvali, [48], has motivated the study of large extra-dimensions, in which gravity might "leak off". The key point of this idea is that the presence of wrapped extra-dimensions could resolve the hierarchy problem. The higher-dimensional Planck scale could indeed be of the same order of magnitude as the electroweak scale if one extra-dimension was of the size of the solar system while all other extra-dimensions would be compactified on a small torus. This extra-dimension would be huge, but if two dimensions were taken to be large, the hierarchy problem could be resolved if their size was only of order of the millimetre.

It is based on this idea that different models where suggested, such as [49] and especially such as the first and second Randall Sundrum models [50, 51]. In these models, (which

 $^{^{3}}$ The Eöt-Wash Short-range experiment has measured the strength of gravity for distances slightly smaller than 0.2 millimetre and have observed no deviations from the Newton's law, but recent experiments are still testing gravity at sub-millimetre scales [46].

we shall describe in great detail in chapter 2), all forces of nature (not counting gravity) are confined on the orbifold branes, where the presence of the orbifold is motivated by heterotic M-theory. This has the advantage of avoiding charges and form fields which are usually present when D-branes are considered. Although this model is greatly simplified compared to any realistic model emerging from string/M-theory, it has the great advantage to be a consistent toy model on which the idea of large-extra dimensions may be tested in cosmology. Later extensions of this model have been proposed to solve the hierarchy problem in more realistic scenarios [52].

A predecessor of the Randall Sundrum scenario was the model suggested by A. Lukas et.al. [53]. After deriving a five-dimensional effective theory of the 11-dimensional Hořava Witten's theory, a five-dimensional cosmological solution was studied and shown to represent a consistent solution for early Universe cosmology. In their model the boundary branes are actually three-brane domain walls on which gauge and matter fields are confined. The Randall Sundrum scenario represents a very special case of their general solution, for which all gauge fields are set to zero.

But the idea that our Universe could be a four-dimensional membrane in a higherdimensional world is much older [54]. One of the first toy models which considered this possibility was given by G. Gibbons and D. Wiltshire in [55]. In that work, the Einstein-Maxwell equations are solved and give a consistent solution, for which the Kaluza Klein corrections may be ignored if the membrane has curvature or if a negative cosmological term is present in the higher-dimensional spacetime. This idea is present in the Randall Sundrum as well, for which a negative cosmological constant is used in the bulk.

1.3 Brane Gas and Dimensionality Puzzle

In the previous section, we showed how the concept of braneworlds could be derived from string theory and in particular, we have suggested the idea that our Universe could be a four-dimensional membrane embedded in a higher-dimensional world. Although this might explain why we only see three spatial dimensions despite the fact that the true world might be higher-dimensional, this does not explain why a four-dimensional membrane was considered in the first place. Unless some specific mechanisms are found to explain why four-dimensional branes are favoured with respect to other dimensional branes, the puzzle remains open. In this section we shall summarise the different ideas that have been proposed to explain why only three spatial dimensions are observed. There have been two main ways of attacking this problem. In the first one, developed well before M-theory was proposed, R. Brandenberger and C. Vafa suggested a reason why only three spatial dimensions could grow large [56]. But the way of attacking the problem has taken another direction since a possible underlying theory with a relatively large extra-dimension, (which would be unobserved in some limit), has been pointed out. Instead of showing why a scenario with only three large spatial dimensions is favoured (or is the only possible scenario), people have tried instead to understand why three-branes (on which we might live, or on which usual physics laws are derived) might be favoured compared to higherdimensional branes. All these ideas rely strongly on the notion of brane gas (as an extension of the string gas concept) which we shall describe first.

1.3.1 Brane gas

Conservative and democratic initial conditions

In the context of string cosmology, the first question one might ask is which initial conditions should we consider. In the standard Hot Big Bang scenario, the Universe is initially extremely hot and dense and hence a homogeneous and isotropic hot gas of matter is usually taken for initial conditions. The same idea may apply here and one might assume the spacetime to be filled with a homogeneous gas of strings. This initial condition is hence "conservative" as argued by the authors of [56, 57]. But since the realisation that D-branes must play an active rôle in string theory, the notion of a gas of strings has been extended to a gas of branes, (where strings are similar to D1-branes, putting aside the issue concerning the brane charge). In this assumption, branes of different dimensionalities are generally considered to be initially present with a comparable density so no brane is favoured initially.

We would like to understand why only three spatial dimensions are observed, or in other words why three spatial dimensions seem to be special and distinguish themselves from the nine spatial dimensions necessary for string theory. For that, one should start with a "democratic" treatment of all the spatial dimensions, ("democratic" as argued again by the authors of [56, 57]). Being compactified or not, all dimensions should be considered exactly the same way in the initial conditions. Hence either all nine dimensions should be uncompactified or all of them should be compactified the same way. In particular, if all dimensions are compactified, a natural assumption would be to take them compactified on a torus of radius $R \sim \sqrt{\alpha'}$. This is a reasonable assumption since the only fundamental scale of string theory is the length scale $\sqrt{\alpha'}$, we might therefore expect all dimensions to be compactified on a torus of this size. This respects the "democratic" condition.

From this initial condition, several proposals have been given to understand how three spatial dimensions might separate themselves out. But first we might raise the question of how this brane gas was created in the first place. In the next subsection will shall therefore follow the argument by [58] where from a configuration including only spacetime filling branes, the creation of a brane gas might be understood. In particular, we shall see how branes of different dimensions might be favoured and hence their density might not be the same.

Creation of D-branes

In order to understand how brane-antibrane pairs might be produced, we shall follow the study of [58]. Working in the framework of superstring theory, with nine uncompactified spatial dimensions, they assume, as a starting point, that all open strings satisfy Neumann boundary conditions. Or in other word, that the space is initially filled by D9-branes and $\overline{D}9$ -branes (antibranes). The initial state presents actually N coincident D9 - $\overline{D}9$ pairs.

This configuration is stable at high-temperature but as the temperature cools down, pairs of D-branes and anti D-branes of lower dimensions are produced. Similarly to a

particle-antiparticle pair, a brane-antibrane pair is indeed an unstable configuration at lowenergy, and the initial state possess tachyons which are related to this instability. At high temperatures, the tachyon is at the minimum of its potential and the configuration is stable, but when the temperature cools down, the potential is deformed and the tachyon is no longer at the minimum but at the maximum. The tachyons hence decay to the true vacuum leading to the annihilation of the D9-D9-brane pairs. But depending on the symmetry group of the vacuum, different regions of the spacetime may take different topological configurations and hence topological defects can be created. Since N coinciding D-branes have an U(N) group, the initial vacuum symmetry group is U(N). We denote by $\pi_k(\mathcal{M})$ the different ways S^k may be mapped on the group \mathcal{M} , $(\pi_k(\mathcal{M})$ is the homotopy group of \mathcal{M}). The different ways S^1 may be mapped onto U(1), for instance, is characterised by the winding number (which is a topological invariant) thus $\pi_1(U(1)) = \mathbb{Z}$. As long as $\pi_k(\mathcal{M})$ is non-trivial (*ie.* there exists different possible maps), the tachyon will take different topological values on different regions of space hence creating a topological defect of codimension (k+1), or a D(d-k-1)-brane, at the intersection of these regions (for a review on topological defects see for instance [59]). In particular, a D3-brane will be created if $\pi_7(U(N)) \neq 0$, which is indeed the case if $N \geq 3$, $(\pi_7(U(N)) = \mathbb{Z})$. With this mechanism, only D-branes with odd spatial dimensions will be created. This mechanism explains how D7, D5, D3 and D1-branes can generically be created out of the initial stable state. But the way they exactly form and the probability for such a creation depends on how many regions need to intersect. Branes of lower dimensions require more regions to have a common intersection (a codimension k object can be seen as the intersection of (k + 1) regions, so the creation of a D3-brane requires seven regions with different topological configurations to intersect, while the creation of a D7-brane requires only three such regions). The creation of D7-branes would hence initially be favoured unless some energetic and entropic arguments are taken into consideration. This will be performed in what follows in order to understand why lower-dimensional branes might be favoured.

However the same mechanism described for the decay of D9-D9 pairs, can now apply for the daughters brane pairs. In particular pairs of D7- $\overline{D7}$ branes may decay and give rise to new topological defects or D-branes of lower dimensions. This will hence create a cascade of daughter branes of lower dimensions.

In this mechanism, the creation of D-branes does not lead, in general, to a homogeneous gas of branes of different dimensions. The initial conditions considered for this scenario may as well appear not very natural. It gives however a consistent mechanism for the production of a brane gas which could be considered in order to understand why only three spatial dimensions are observed.

1.3.2 String theory contributions to the dimensionality puzzle

Annihilation of extended objects

We shall now focus on the "dimensionality puzzle", trying to understand either how three dimensions may distinguish themselves or whether D3-branes are favoured in some scenarios.

Both these ways of attacking the problem rely on the similar idea that strings or other extended objects may not annihilate for large dimensions but can for lower dimensions. In order to understand this, we may use the analogy with monopoles, comic strings and domain walls in standard four-dimensional cosmology. As already mentioned, in three infinite spatial dimensions, the probability for two monopoles to "collide" is very weak and in general we consider that monopoles do not decay, thus the magnetic monopole puzzle. Two cosmic strings, on the other side, (unless they are parallel to each other) will interact and can annihilate. The same will hold for any extended object of dimension p such that $2(p + 1) \ge (d + 1)$, where d is the spacetime spatial dimension. When there is equality, 2(p + 1) = (d + 1), such as for strings in d = 3 dimensions, the collision will happen only at an instant, but for 2(p + 1) > (d + 1), the intersection happens at all time, this is for instance the case for domain walls (with p = 2), their intersection can be seen as a 1-brane (or string). The same will be true in higher dimensions, we thus expect extended objects with p spatial dimensions (or p-branes) with 2p < (d-1) to survive in a (d+1)-dimensional spacetime. For larger objects, on the other hand, they will interact and annihilate.

Brandenberger Vafa mechanism

Long before the idea of branes had gained much interest, R. Brandenberger and C. Vafa suggested a proposal [56] to understand why only three large spatial dimensions can be large. Their starting point was the assumption that at the beginning all nine spatial dimensions were small and probably compactified on a very small torus of radius $R \sim \sqrt{\alpha'}$. Strings will then naturally wind around this torus, preventing the dimensions to expand since the energy required for a winding mode would grow as R/α' . Spontaneously, however some dimensions can tunnel to a state where they grow a bit larger. When this happens, strings wrapping these dimensions compress them back to their normal size; or this is what happens in the usual case. But if three or less spatial dimensions grow large, strings wrapping these dimensions may annihilate and let these dimensions "free", as can been seen in fig. 1.4. Indeed, if we consider strings (extended objects with p = 1), in



Figure 1.4: Two winding and antiwinding modes can intersect in three or less dimensions and form a loop (plus radiation).

dimensions lower that $d \leq 2p + 1 = 3$ they may interact with each other and hence annihilate. This is not the case, however, for dimensions greater than three, and is hence an argument explaining why no more than three spatial dimensions may grow large. Of course, only two or one dimension might have grown large, but if we imagine a scenario where the small compactified dimensions may spontaneously grow large with a non vanishing probability, there would be as many dimensions growing large as possible, hence three dimensions. Another argument relying on the anthropic idea has already been mentioned. In this mechanism, the other dimensions will remain small and "compressed" by the strings winding around them. The pressure applied by the strings on the compressed dimensions might even help the expansion of the other ones.

In this scenario, although some dimensions are capable of expanding and becoming large, they will still remain compactified. For this to be consistent with present observations, the radius of their present compactification must be larger than the Hubble radius. In such a case, we would not be able to notice this compactification and our Universe would effectively appear to have three infinite dimensions.

The notion of T-duality can be extended to branes [57] and in this mechanism, one can imagine instead that a brane gas is filling the spacetime and wrapping compactified dimensions. The argument will then be exactly the same since strings are the extended objects with the least dimensions, they can still prevent more than three dimensions to expand while higher-dimensional branes can annihilate.

In the context of heterotic M-theory, the nine-dimensional compactified spacetime described above could play the rôle of the orbifold fixed point of the \mathbb{Z}_2 symmetry imposed on the 11th dimension. If only three out of the nine dimensions were allowed to expand, in heterotic M-theory this would effectively correspond to the presence of a three-dimensional braneworld (at the orbifold fixed point) embedded in a five-dimensional spacetime.

Early simulation works have confirmed the hypothesis of this model [60]. However more recent work seem to suggest that this mechanism can indeed happen but its probability appears to be weak [61].

Majumdar Davis mechanism

Since the appearance of branes as a "popular" object of string theory, another way of explaining the puzzle has been proposed. Although different alternatives to the problem have been suggested the main idea behind them is similar. Working in the framework of superstring theory (in particular we will choose type IIB), with nine uncompactified spatial dimensions, the different proposals rely on the same idea that at some point the entire spacetime could have been filled with a gas of branes and antibranes, but the different scenarios suggest slightly different initial conditions and evolutions. Nonetheless, they all have the final conclusion that D3-branes are favoured compared to other (higher-)dimensional branes, although we should as well point out that D7-branes play an important rôle.

We have seen earlier how the study of [58] could explain the creation of a brane gas from the decay of pairs of D9-D9-branes. In that mechanism, branes of lower-dimensions were however not necessarily favoured in comparison to the higher dimension ones. In order to explain how D3-branes might be favoured, one needs to rely on energetic and entropic arguments as is done in [58]. To perform this last step, they compute the entropy of a gas of D*p*-branes and show that it goes as p^{-1} . They can then conclude that the probability of ending in such a configuration is exponentially suppressed compared to the probability of ending in the same configuration with instead a gas of D*q*-branes with q < p. Lower-dimensional branes are hence favoured as long as the initial state possess a large number N of D9-D9 pairs.

In later work [62], they consider the annihilation of brane-antibrane pairs. Although the creation of higher-dimensional branes is favoured in comparison to lower ones, their calculation showed that D-branes of higher dimensions annihilate more rapidly than lowerdimensional ones. Starting with a gas of branes of different dimensionality, at late times, the spacetime might be filled with only lower-dimensional ones. It is relying on this last idea that the proposals in the next subsection have been suggested.

Annihilation of brane-antibrane pairs

The previous argument, relying on the annihilation of D-D-brane pairs, is at the origin of new proposals [63, 64]. In that case, a gas of D-branes and antibranes of arbitrary dimensions is directly considered as the initial condition mentioned earlier. Spontaneously, pairs of branes and antibranes can then annihilate and using a simple dimensional argument, only branes of dimension three or lower are expected to survive. This is a simple consequence of the fact that in a (d+1) = 10-dimensional spacetime, only extended objects of dimension $p \leq (d-1)/2$ can miss each other and hence survive. The other D*p*-branes will indeed either intersect at all times or interact at one point but will eventually disappear, creating at the same time the entropy of our Universe.

This is the proposition given by [63] and then extended by [64] to expanding spacetimes. In this last paper, an argument relying on the energy density of the different Dp-branes is used in order to select the most probable configuration in an expanding ten-dimensional spacetime. In their proposal, the Dp-brane with the highest energy-density is assumed to be the most favourable one. They then compute the energy density of the different Dpbranes in an expanding Universe and point out that the total energy density is dominated by the D3-branes and D7-branes, which are hence the most favourable branes.

The argument goes as follows: If the spacetime is filled with a (homogeneous) fluid with equation of state $p = \omega \rho$ where p is its pressure and ρ its energy density, the scale factor a will scale as $a \sim t^{2/d(\omega+1)}$, where t is the proper time and d the spacetime dimension (d = 9). For $\omega < \omega_c = -1 + 2/d$, the expansion accelerates, we will hence consider only cases for which $\omega \geq \omega_c$. A homogeneous distribution of D*p*-branes, that do not interact, can be considered as a fluid with $\omega_p = -p/d$ and energy density $\rho_p \sim a^{p-d}$. If the D*p*-branes interact, *ie.* for p > 3, their energy density scales instead with the Hubble parameter and not the scale factor: $\rho_p \sim (H^{-1})^{p-d} \sim t^{p-d}$. If no branes where interacting, the higher-dimensional branes will hence be favoured in an expanding spacetime since their worldvolume is considerably larger, but interactions modify this argument. If at some point spacetime is dominated by D3-branes, effectively the spacetime would be filled with a fluid with $\omega = -1/3$. The energy density of the D3-branes will hence scale as $\rho_3 \sim a^{-6} \sim t^{-2}$. Among the interacting branes, the only ones which would have a comparable energy density are the D7-branes: $\rho_7 \sim t^{-2}$ and the higher-dimensional ones. However, one expects the D8-branes not to be present in type IIB string theory (which only contains branes with odd spatial dimensions). The D9-branes are different since they are space filling and overlap entirely, the authors of [64] hence argue that their energy density is zero. The D3-branes together with the D7-branes are therefore favoured in that situation.

Even if the spacetime is initially not dominated by D3-branes, $\omega > -1/3$, the D3branes will still have the greatest energy density (this time even compared to D7-branes), and the spacetime will eventually be filled with mainly D3-branes. Their initial assumption is hence reasonable and gives a very simple argument explaining why D3 and D7-branes are the branes dominating the spacetime energy density.

These arguments only give a rough proposal on how one might attack the problem and explain why we are living in (3 + 1)-dimensions, but at least they suggest that a possible solution to the puzzle might be found in the future. Several issues should be studied in more detail, such as what would happen with the last mechanism if some dimensions were compactified, and if branes interactions were considered in a more general way. But for the purpose of this thesis we will concentrate on branes with only three spatial dimensions and motive this choice by the previous arguments.

1.4 Brane Inflation

Motivated by the arguments of the previous section, we now study the contribution of three-branes to cosmology. For this, we only give a brief overview of the leading models which have first been proposed in the context of brane inflation or its alternative. This will not be an exhaustive presentation as we will only concentrate on specific models, which have been greatly extended since.

1.4.1 Dilaton and Moduli Fields

One of the most interesting features of string theory is its scalar fields abundance which arises naturally without the need for any specific assumptions. This seems to be a very different situation from other areas of physics where scalar fields are assumed to be present, but their origin can not be explained. The presence of these scalar fields in string theory can for instance be associated with the presence of compactified dimensions. The moduli fields are indeed related to the size of these dimensions. Unless some non-perturbative effects are considered, the compactification radius is free to vary. In terms of the moduli fields, this imply that their potential should be flat. In string theory, the strength of the gravitational coupling may be freely redefined. Another scalar field, the dilaton, can thus be associated with this degree of freedom. And one should expect its potential to be as well flat. The dilaton is indeed a massless scalar field. String theory hence predicts the existence of massless moduli and dilaton scalar fields. Their existence is at first puzzling since no massless scalar fields are observed. But if supersymmetry is broken, their potential is expected to be modified and such fields might becoming massive. Such supersymmetry breaking could be generated by non-perturbative effects. One of the most important issues in this subject is the necessity of stabilising these fields. Much effort has indeed been devoted to the study of realistic models capable of generating an effective potential for the dilaton, with a minimum where the dilaton may sit. On the other hand, the flatness of the moduli and dilaton field potentials could have a very interesting application. A flat potential is indeed one of the most important requirements of inflation. Dilaton and moduli fields can thus be seen as very good candidates for the inflaton scalar fields [65], this is very interesting since the origin of the scalar field in inflation remains an open question.

1.4.2 D-brane Inflation

One of the first proposals that has made use of the notion of D-branes to cosmology is the one suggested by G. Dvali and S. Tye in [66]. Their idea is to derive inflation from a model including two D-branes. They first consider two coinciding D3-branes embedded in a bulk with N large extra-dimensions (and suppose that $N \ge 2$, hence giving a reasonable explanation to the hierarchy problem). Calculating the forces between the branes, they show that this configuration is stable. However, this might not be the initial conditions, and to start with, they consider the branes to be separated by a distance r while they still remain parallel. As mentioned previously, one of the most interesting contribution of D-branes to cosmology could be the interpretation of this moduli field (the distance r or the size of the extra-dimension) as playing the rôle of the inflaton scalar field.

The supersymmetric configuration will be BPS (Bogomolny-Prasad-Sommerfeld [67, 68]), if the contribution from the different forces between the branes cancels out and leads to a static and stable configuration where the branes are on top of each other. The different interaction sources between both D-branes are the followings:

- The modes localised on the branes do not couple when the branes are largely separated. When they get close to each other $r \to 0$, on the other hand, these modes couple and give a negative contribution to the vacuum energy of the brane which cancels with the positive branes tensions. When the branes are close, this contribution vanishes, but when they are separated, this gives a contribution to the potential $V(r) = 2\lambda$ where λ is the branes' tension.
- As already mentioned, D-branes have charges (Ramond-Ramond charges) which will create a repulsive force between the branes and hence give a negative contribution to the potential $V(r) \sim -2r^{2-N}$.
- Dilaton closed strings can as well be exchanged between both D-branes and contribute positively to the potential $V(r) \sim r^{2-N}$.
- Finally, the exchange of graviton closed strings gives rise to a gravitational attraction between the branes V(r) ∼ r^{2−N}.

The three last interaction terms therefore cancel in the supersymmetric BPS configuration and since the first term vanishes when the branes are close (or on the top of each other), this leads to a static configuration at r = 0. But when non-perturbative effects are generated, supersymmetry could be broken. The bulk modes will get massive and create instead a Yukawa potential of the form $r^{2-N}e^{-mr}$, where m is their effective mass after supersymmetry breaking. This affects both the dilaton strings and the charge exchange and the effective potential between the branes for large brane separation hence becomes:

$$V(r) \sim 2\lambda + \alpha r^{N-2} \left(1 + e^{-m_{\Phi}r} - 2e^{-m_{RR}r} \right), \qquad (1.10)$$

where α is a constant and m_{Φ} , m_{RR} are the different fields masses after supersymmetry breaking. Although the exact details of this calculation are very hard (if not impossible) to calculate, this suggest a mechanism which could lead to a very flat potential (1.10) in which the inflaton scalar field (associated to r) could evolve and create a scale invariant power spectrum. Another beautiful aspect of this model is its graceful exit from inflation, which would naturally happen when the branes get close to each other. When $r \to 0$, the contribution of the first term is modified and we should include the contribution of the localised modes on the branes which vanish exponentially with r. For small r, this will lead to an exponential contribution to the potential which could hence be steep, inflation would stop and the reheating period will start.

1.4.3 Brane - Antibrane Inflation

Despite the elegance of the previous scenario, its lack of proper non-perturbative calculation irrefutably represents a heavy down side. This weakness can be cured if one considers its natural extension using instead a non coincident brane-antibrane pair [69–71]. Although the work in [69] takes for initial conditions a set of non-coincident parallel D-branes and antibranes, the main results remain the same. In [70], it is shown that the different interaction sources from the previous D-brane pair model remain the same although the interaction between the charges on each branes generates now an attractive force. Hence the charge contribution does not cancel out the contribution from the dilaton and graviton exchange in the BPS configuration. In that case, a simple naïve calculation in perturbative theory can be performed and gives rise to a Newtonian potential (plus a positive constant). Thanks to the presence of the positive constant, the potential will prevent the branes from separating too much, and hence avoid a "runaway" form. Unfortunately, the slow-roll conditions required for inflation to work are not satisfied (although the potential is flat, its second derivative is not small enough unless the distance r between the branes is taken to be much larger than the size of the dimensions perpendicular to the brane, which is not possible). In order to remedy this problem, a more elaborate calculation should be made. In particular, if the large extra-dimensions (perpendicular to the branes) are compactified on a torus, one should include in the calculation the different images of the branes in the covering space of the torus (especially if the distance between the branes is comparable to the torus radius). When this is taken into consideration, one gets instead a potential which is quartic in the brane separation r and hence satisfies the slow-roll conditions required for inflation. This is, to our knowledge, the first model capable of giving an origin for the inflation potential and exactly computing its form from a purely string theory scenario.

Furthermore, it has the very nice feature of giving a computable scenario for the end of inflation. If one considers open strings to stretch between the branes, their mass depends on the exact distance r. The interesting point is that the mass vanishes for a (finite) critical brane separation and becomes tachyonic below the critical value. This leads to a tachyonic scalar field which contributes to the potential. The potential becomes a function of two scalar fields (the tachyon and the distance between the branes characterised by r) and is actually of the same form as the one usually proposed in hybrid inflation in order to facilitate the end of inflation. The sudden appearance of a tachyonic scalar field when the branes are close, makes it possible for the scalar field r to "leak" or transfer part of its energy and hence provide a graceful (and computable) exit for the end of inflation [72].

This model has inspired many works relying on the same idea and extending the original scenario [73] (where the assumption of parallel branes is, for instance, relaxed [74,75], or branes could be of different dimensionality, such as in a model of D3/D7-brane inflation [76] or D4/D6 [77]). For a review on brane inflation, see for instance [78,79]. One interesting idea, connecting with notions mentioned in the previous section, is the generalisation of this model to brane gas. It is interesting to point out that the tachyonic part of the potential can lead to topological defects (and hence D*p*-branes by the same mechanism as expressed in section 1.3.1). The spacetime could hence be filled with a brane gas. The same D-brane inflationary scenario could then apply to pairs of branes which are not coincident [80].

1.4.4 Inflation on a Probe D3-brane driven by a DBI Action

Another scenario worth mentioning is the one presented in [81]. In that model, the brane velocities could be large. As we shall see, this is an issue to which a very special interest will be given in this thesis. They consider N coinciding D3-branes, where all moduli fields have a vanishing vacuum expectation value (VEV). Then they study the possibility of one brane to peel off (corresponding to one moduli field ϕ having a non-vanishing VEV $\langle \phi \rangle \neq 0$). At low-energy or low-velocity, the kinetic contribution arising from the motion of this brane is given by the standard one $\mathscr{L}_k = -\frac{1}{2}v^2$, where v is the brane proper velocity and is associated to the moduli field evolution by $v = \sqrt{\lambda}\dot{\phi}/\phi^2$, λ being the coupling constant. But this can be shown to be only the low-velocity leading term arising from the exact Dirac Born Infeld (DBI) Lagrangian of a probe D3-brane (whose exact form may be found in [82]). Ignoring for simplicity the coupling to gravity, the DBI Lagrangian for the probe D3-brane can be expressed as:

$$\mathscr{L}_{\text{DBI}} = f(\phi)^{-1} \left[\det \left(-\eta_{ab} - f(\phi) \partial_a \phi \, \partial_b \phi \right)^{1/2} - 1 \right], \tag{1.11}$$

where $f(\phi) = \lambda/\phi^4$. At low-velocity, we indeed recover the usual kinetic contribution, but beyond this regime, higher powers in v should play a significant rôle. We may as well notice that in that case the velocity is bounded v < 1, which is nothing else but the fivedimensional Einstein causal speed limit and the parameter $\gamma = (1 - v^2)^{-1/2}$ is analogous to the special relativistic factor (this speed limit on the moduli field has already been pointed out in [83]).

We may as well point out that the potential term -1 in the action cancels with the leading term of the square root expansion. The brane is hence BPS.

The down side of this model is the introduction of a potential for the moduli field ϕ capable of generating inflation. In this model its origin is not explicitly calculated although some arguments are given. The presence of $\overline{D}3$ -branes in the bulk, for instance, might contribute with a quartic potential. This is a simple consequence of the fact that the DBI Lagrangian of the antibrane can be expressed just as (1.11), with the potential term being +1 instead of -1. The leading term in the square root expansion does not cancel

and contributes with a term of the form ϕ^4/λ . Some other contributions to the potential for the moduli field are as well to be expected when the coupling to gravity is taken into consideration or when interactions with other fields are considered.

Once such a potential is generated, an explicit calculation for the power spectrum of density perturbations is given and is shown to correspond to what would be predicted by usual four-dimensional inflation, despite the fact that no constraint is assumed on the probe-brane velocity (or equivalently on the moduli evolution). The ratio between scalar perturbations and tensor perturbations is shown to be decreased by a factor $\sqrt{1-v^2}$. Measurement of tensor perturbations might hence give some constraints on v.

In this model, the potential can not be explicitly derived, and constraints on the potential are hence just imposed by hand. Despite this fact, this model is of great interest as it raises the possibility to extend the notion of inflation to a regime where the slow-roll conditions do not necessarily imply a slow evolution of the scalar field when higher derivatives corrections to the initial low-energy model are present. The study of higher-derivative corrections and their impact on cosmology (and especially to inflation) has as well for instance been studied in [84,85]. In these models, an inflationary expansion of the Universe can be driven by these higher-order derivative terms, even without the introduction of a potential. However an origin for these higher-derivative terms is not explained. It is hence interesting to study this proposal in the context of braneworld models, which generically produce such corrections.

1.4.5 Ekpyrotic Scenarios

We finish this section on brane inflation by giving a brief overview of the Ekpyrotic scenarios [86,87] which share with the previous model the possibility of generating a scale invariant spectrum when slow-roll conditions are violated. Based on heterotic M-theory, both Ekpyrotic scenarios consider the 11th dimension to be compactified on a S^1/\mathbb{Z}_2 -orbifold with two orbifold branes on the fixed point of the symmetry. Relying on the analysis of section 1.3.2, one can consider the fixed point to have only three large or uncompactified dimensions, and will hence be considered as orbifold three-branes. One of the brane is supposed to have a positive tension, while the other one has a negative one.

In the first Ekpyrotic scenario [86], the authors consider the possibility for a bulk brane to peel off from the positive tension brane and propagate in the bulk until it collides with the negative tension brane. The different interactions between the branes and possible terms arising from bulk degree of freedom is modelled by the introduction of a potential for the distance between the branes. For adequate constraints on the potential, a scale-invariant power spectrum might be produced [88]. But one of the most interesting realisations of this model is that the collision of the bulk brane to the boundary brane will create a fourdimensional singularity (although from the higher-dimensional point of view, the geometry remains regular). This can hence be interpreted as the origin of the Big Bang singularity (giving the name Ekpyrotic to the scenario, which in the Stoic philosophy describes a Universe which is consumed by fire and gets recreated from the same fire).

An interesting alternative to this model has been given in a second Ekpyrotic scenario [87], where the rôle played by the bulk brane is suppressed. Instead the orbifold branes are directly studied in a regime where they move towards each other. In that case, the collision of the two orbifold branes is considered as the origin of the four-dimensional Big Bang. The singularity is this time five-dimensional and makes the study of physics through the collision very hard to follow. These models offer the very interesting possibility of the generation of the large scale structure before the Big Bang, as opposed to after the Big Bang as usually proposed in inflationary models (although some models such as the Pre Big Bang model have explored the same idea [89]). The down side of this model, is the necessity of transmitting perturbations through the singularity which is a very difficult and contestable task to achieve [90, 91].

After the boundary brane collision or Big Bang, the branes move apart for a long period until the potential becomes slightly positive and stops the expansion of the 11th dimension. The branes are thus forced to move back towards each other again and the same process starts again. This leads to an eternal and cyclic model of the Universe.

1.5 Outline

So far a very brief overview has been given. This introduction does not pretend to reflect an objective nor exhaustive view on the subject (which is subject to much polemics itself), but rather to present different concepts I have personally came across throughout my PhD course and have found especially interesting. I hope this might help understanding why so much interest and effort has recently been given to braneworld models and how this area can be tightly linked to cosmology. It indeed offers an entire new set of possibilities for the study of cosmological scenarios and incredible progress have been made in the past six years. As a very small contribution to the study, I have focused my thesis on the understanding of braneworld cosmology in regimes beyond the low-energy limit. I believe these regimes to be of interest as it is beyond the low-energy limit that braneworld models differentiate themselves the most from other more "standard" cosmological models.

From an effective field theory point of view, one generally tends to incorporate the higher-dimensional degrees of freedom (which come in at high energies) by integrating them out. The resulting theory thus contains additional higher derivative terms (which are generally negligible) in the four-dimensional effective action. However this procedure will only be valid when the energy scales of the modes which have been integrated out are large in comparison to the four-dimensional ones. Beyond the low-energy regime (*ie.* when the four-dimensional energy scales become comparable to the one which have been integrated out), this four-dimensional effective field theory approach should hence break down.

In order to work beyond the low-energy approximation, I suggest instead to attack the problem with a different perspective. After studying the five-dimensional effects, I suggest to translate them into a four-dimensional language. This is in general a very challenging task to do, but once some approximations are made for some specific cases, much progress can be made. The aim of this thesis is hence to explain this way of approaching the problem and to illustrate it with some specific examples. For this I suggest to proceed as follows:

In chapter 2, I give a more detailed study of braneworld cosmology. In particular I focus

on the study of the modified Einstein equation on a brane, its low-energy limit and departures from usual four-dimensional gravity. In order to simplify the description, I focus on a remarkably simple toy model: the Randall Sundrum model. Extensions to more realistic models are then discussed.

The physics presented in this section is an overview from prior studies and developments. However the derivations of many of the results has been expressed in a more original form which will connect more easily with further works presented in the next chapters.

- In chapter 3, a formalism is presented for the study of brane inflation in a regime where corrections to standard four-dimensional theories are present. These corrections, are first pointed out in chapter 2, but take an integral part of chapter 3. I present a formalism in a general setup and then focus on the study of the one-brane limit of the Randall Sundrum model. In that special limit, the evolution of an inflaton scalar field on the brane is studied. Making use of perturbation theory, the power spectrum and the spectral index for both scalar and tensor fluctuations are computed, and the magnitude of non-gaussianity perturbations can be estimated. Departures from standard four-dimensional inflation can be stressed, although very few differences can actually provide us with a systematic way to distinguish inflation occurring in a genuinely four-dimensional spacetime and inflation on a brane embedded in a higherdimensional world. This is not surprising as corrections arising from Kaluza Klein modes have been neglected in that study.
- The aim of chapter 4 is hence to work beyond this assumption. Although Kaluza Klein modes can be properly computed in only very few specific models, their study is still precious since it enables us to understand how Kaluza Klein modes behave in these regimes. It is then possible to model them in a four-dimensional formalism and to use this model on a larger class of scenarios. This is exactly what is done for the first Kaluza Klein mode in the quasi-static limit. After suggesting an ansatz that correctly models the first Kaluza Klein mode in the quasi-static case, I extend the study to the

previous model of brane inflation. This provides a way to track part of the Kaluza Klein contribution, although the exact behaviour can be much more complicated. In particular, I study the effects this term has on the production of scalar and tensor perturbations. These are shown to propagate at a sound speed slightly different from the speed of light and the estimate of the non-gaussianity suggests a possible deviation from usual inflation predictions. I then use this formalism to test the duality between slow-roll and fast-roll potentials and show that corrections arising from Kaluza Klein mode will generically break the duality.

These last two chapters are mainly based on work which has been summarised in [92] and some unpublished works which I have found interesting have been included.

A different route is then taken in chapter 5, where I focus instead on the study of the Randall Sundrum model in the close-brane limit. In that limit, I develop a four-dimensional effective theory which is checked to be consistent with previous studies and gives sensible results in different limits. This new theory presents higherderivative corrections which can now be studied in a new and original way. As practical and non-exhaustive examples, I suggest to show how this theory may be applied to the study of perturbations in different scenarios. In the first one I consider fluctuations generated by a stiff source of matter present on the brane. I show that perturbations evolve in a similar way than in genuinely four-dimensional theories, but I point out that the effective Newtonian constant is affected. As another example, I study the propagation of perturbations when a potential for the radion is introduced. In that model, I show that the behaviour is very similar to standard four-dimensional studies. Unfortunately, this does not provide a way to clearly distinguish braneworld scenarios from four-dimensional ones, but I believe that the effective theory which has been developed in this chapter can be extended and used to work with more realistic models.

This chapter is strongly based on work in collaboration with Sam Webster [93], although it has been presented in a way which reflects my own contribution and point of view to the study. The last two sections include unpublished personal work.

Finally, the results are summarised in chapter 6, where the key points of this thesis are stressed. Despite some critics that I shall present, many possible extensions can be studied which I believe to be worth some interest.

Chapter 2

Braneworld Cosmology



Shu, Holder of the Sky (Egypt)

2.1 Introduction

In all areas of physics, the study of a simple toy model can allow us to understand many of the central features of a specific problem. If we really live on a brane, be it a D-brane, an orbifold brane, or any other kind of brane, the first question one might address is how would we know about it, or more physically, what would be the key observational signatures that would indicate that this is indeed the case? To understand these characteristic braneworld properties, we will concentrate on the study of a remarkably simple toy model: the Randall Sundrum (RS) model [50,51]. As we shall see, the underlying idea behind this scenario is relatively straight-forward, nevertheless, the physics is far from trivial. Several features of

braneworld cosmology can be easily understood through the study of this toy model. In particular, one of the main focus of this thesis is to understand the effect of five-dimensional physics on our four-dimensional world assuming we are living on a brane. To perform this task, we need a covariant formalism capable of describing the theory on the brane. This is possible through the Gauss-Codacci formalism which projects the five-dimensional bulk equation onto the branes. We present this formalism in this chapter and show how, applying this idea to the RS model, brane gravity differs from ordinary four-dimensional gravity. The main departure from four-dimensional gravity comes from the coupling to quadratic terms in the stress-energy and from the presence of an additional tensor (the induced five-dimensional Weyl tensor) which encodes information on the bulk excitations. Due to the presence of this term, which is not determined from a four-dimensional point of view, the Einstein equation is therefore not enough to study the brane cosmological evolution, since the system is not closed. These key features are more easily understood for the special case where cosmological symmetry is considered. For that specific case, the fivedimensional geometry is remarkably simple, and the five-dimensional effects on the brane can be pointed out very clearly. However if we want to make further analytical progress without assuming specific symmetries, one should understand the contribution of this Weyl tensor by studying its evolution in the bulk. The derivation of its evolution is presented in this chapter and in particular we show how it is possible to extract information from this now closed-set of equations in order to obtain an effective theory on the brane. This is in general very subtle, but by making use of a low-energy approximation, remarkable progress can be made. The resulting theory on the brane is nothing but standard gravity coupled to a scalar field, which can be interpreted as a moduli representing the distance between the branes. However this four-dimensional low-energy effective theory misses out all the key features that distinguish braneworld gravity to standard four-dimensional Einstein gravity (or its Brans-Dicke extension [94–96]). Although it is reassuring to recover four-dimensional gravity in some limit, working in this regime does not allow us to point out any peculiar features which could not be explained from a four-dimensional theory of gravity and therefore could provide a signature of braneworld cosmology. Instead, we would like to track down specific features of braneworld cosmology which could be used to differentiate both models and hopefully could be checked by observations. For this, it is essential to work beyond the low-energy approximation. In that regime, the bulk excitations are visible on the brane and give rise to Kaluza Klein modes, which are generic to higher-dimensional theory. To understand their behaviour and how it might be possible to model them, we first present their effects for static branes. In that case, it is possible to solve the five-dimensional equations at the perturbed level and hence, to see the effect of the Kaluza Klein modes in that regime. We shall then see in chapter 4, how it might be possible to model some of their behaviour.

Finally we present possible extensions of the RS model to more realistic braneworld models. This can be done either by introducing bulk scalar fields or by considering bulk branes. For late time cosmology, a realistic braneworld model should have its moduli stabilised which is not the case for the RS model. Moduli stabilisation can be facilitated by the presence of bulk scalar fields, and although this is not the main focus of this thesis, we will give a brief overview of this issue. Then we concentrate on the study of bulk branes, for which gravity couples to matter in a very peculiar way and is therefore of great interest for the purpose of this thesis. But first we shall start by giving a description of the RS model on which most of this thesis will focus.

2.2 Description of the Randall Sundrum model

In what follows, we shall be interested in the two brane RS model as a specific simple example of braneworld cosmologies. For a review of braneworld cosmology, see for instance [97]. In that model, the spacetime is five dimensional, with a compact extra-dimension having the topology of an S_1/\mathbb{Z}_2 orbifold. The stress energy of the bulk is assumed to be from a pure negative cosmological constant $|\Lambda| = \frac{6}{\kappa L^2}$, $\kappa = \frac{1}{M_5^3} \equiv \frac{L}{M_4^2}$, with M_n the n-dimensional Planck mass. Two boundary branes with positive and negative tension (referred as \pm -branes) are assumed to be located at the fixed points of the \mathbb{Z}_2 symmetry on which gauge and matter fields with Lagrangians \mathscr{L}_{\pm} are confined [50]. The presence of the negative cosmological constant in the bulk has the effect of curving the bulk geometry, warping the fifth dimension and hence making gravity on the positive tension brane more easily localised than in a usual compactification à la Kaluza Klein. In the original RS model, which was proposed as a possible explanation for the hierarchy problem, our Universe was suggested to lie on the negative tension brane. Because gravity is more easily localised on the positive tension brane and because of the way gravity couples to matter on the negative tension brane, in this thesis we will be more inclined to assume that our Universe lies on the positive tension brane.

In the original two brane RS scenario [50], although no stabilisation mechanism is present, the branes are static and the distance d between the branes is arbitrary. Since no stabilisation mechanism is present, this configuration is unstable and one of the aims of this thesis will be to understand the effects of the variation of d on the branes geometries. The second RS scenario [51], can be seen as the limit of this first model, when the distance between the branes is taken to infinity. In this limit, the fifth dimension is infinite and has a \mathbb{Z}_2 topology with the positive tension brane on the fixed point of the symmetry. Gravity on the positive tension brane indeed decouples from the negative tension brane and, at large distances, gravity on the brane satisfies Newton's force law.

It is in this limit that we shall first derive the induced equations on the one-brane model in the next subsection, its generalisation to the two-brane case will be straightforward.

2.3 Gauss-Codacci Formalism

2.3.1 Derivation of the Gauss-Codacci Equations

In order to derive the induced equations on a RS brane, we consider the five-dimensional bulk with negative cosmological constant Λ and a four-dimensional hypersurface \mathcal{M} on which matter fields are localised. This hypersurface has as well a tension λ and we denote by $\mathscr{L} = \mathscr{L}_{matter} - \lambda$ the total Lagrangian on \mathcal{M} . For simplicity, we choose coordinates



2.3. GAUSS-CODACCI FORMALISM

Figure 2.1: Two brane Randall Sundrum model.

where the metric is of the form

$$ds^{2} = g_{ab}dx^{a}dx^{b}$$

= $A^{2}(y,x)dy^{2} + q_{\mu\nu}(y,x)dx^{\mu}dx^{\nu},$ (2.1)

with x^{μ} the transverse coordinates and y parameterising the extra dimension. We choose this frame, such that the hypersurface \mathcal{M} is located at the fixed position y = 0. Such a gauge can indeed always be fixed. If \mathcal{M} was not fixed at y = 0, but was moving with location $y = \bar{y}(x^{\mu})$, we could perform the coordinate transformation $\tilde{y} = y - \bar{y}(x^{\mu})$ such that \mathcal{M} would be located at $\tilde{y} = 0$. The metric would then be of the form $ds^2 =$ $A^2 (d\tilde{y} + \bar{y}_{,\mu} dx^{\mu})^2 + q_{\mu\nu}(y, x) dx^{\mu} dx^{\nu}$, and the cross term $d\tilde{y} dx^{\mu}$ could be cancelled by a suitable change of the x^{μ} coordinate. It is therefore always possible to find a frame of the form (2.1) for which the hypersurface \mathcal{M} is at fixed position [98–100]. In that frame, $q_{\mu\nu}(\bar{y}, x)$ is the induced metric of a $y = \bar{y} = \text{const}$ hypersurface. The Einstein tensor on a y = const hypersurface will be written simply as $G_{\mu\nu}$, whereas the fully five-dimensional tensor will be denoted by ${}^{(5)}G_{ab}$, and similarly for any other tensor. We use the index conventions that Greek indices are four dimensional, labelling the transverse x^{μ} directions, while Roman indices are fully five dimensional. The symmetrisation over two indices will be denoted with a bracket $Z_{(\mu\nu)} = (Z_{\mu\nu} + Z_{\nu\mu})/2$, whereas a square bracket will represent the antisymmetric part $Z_{[\mu\nu]} = (Z_{\mu\nu} - Z_{\nu\mu})/2$. We denote by ∇_{μ} five-dimensional covariant derivatives with respect to g_{ab} and D_{μ} four-dimensional ones with respect to $q_{\mu\nu}$. In this subsection, we give a brief overview of the derivation of the Gauss-Codacci equations and the Israël matching conditions on the brane which, as will shall say later, will play a similar rôle as the Einstein equation and the conservation of energy constraint on the hypersurface.

Our starting point shall be the five-dimensional action [101]:

$$S = \int d^5x \sqrt{-g} \left(\frac{1}{2\kappa} {}^{(5)}R - \Lambda \right) + \int_{\mathcal{M}} d^4x \sqrt{-q} \left(\mathscr{L} - K \right).$$
(2.2)

For a y = const hypersurface we denote by $K_{\mu\nu}$ its extrinsic curvature and n_{μ} its unit normal vector. K is the trace of $K_{\mu\nu}$: $K = q^{\mu\nu}K_{\mu\nu}$ and $K_{\mu\nu}$ is defined by [101, 102]:

$$K_{\mu\nu} = \frac{1}{2} \mathcal{L}_n q_{\mu\nu} = -\frac{1}{2} n^a \nabla_a q_{\mu\nu} + q_{a(\mu} D_{\nu)} n^a, \qquad (2.3)$$

where \mathcal{L}_n is the Lie derivative with respect to n. For a purely four-dimensional tensor, $\mathcal{L}_n = A^{-1}\partial_y$.

We may define the electric part of the five-dimensional Weyl tensor $E_{\mu\nu}$ as:

$$E_{\mu\nu} \equiv {}^{(5)}C^a_{\ bcd} n_a n^c q^b_{\ \mu} q^d_{\ \nu}.$$
(2.4)

From the properties of the five-dimensional Weyl tensor ${}^{(5)}C^a_{bcd}$, we can easily check that $E_{\mu\nu}$ is traceless with respect to $q_{\mu\nu}$. In the frame (2.1), using the five-dimensional Einstein equation $G_{ab} = \kappa \Lambda g_{ab}$, the expression for $E_{\mu\nu}$ is

$$E^{\mu}_{\nu} = {}^{(5)}C^{y}_{\ \alpha y\nu}q^{\alpha \mu} = {}^{(5)}R^{y}_{\ \alpha y\nu}q^{\alpha \mu} + \frac{1}{L^{2}}\delta^{\mu}_{\nu}$$
$$= -\mathcal{L}_{n}K^{\mu}_{\nu} - A^{-1}D^{\mu}D_{\nu}A - K^{\mu}_{\alpha}K^{\alpha}_{\nu} + \frac{1}{L^{2}}\delta^{\mu}_{\nu}, \qquad (2.5)$$

where we used the expression for the Christoffel symbols: $\Gamma^{y}_{\mu\nu} = -A^{-1}K_{\mu\nu}$ and $\Gamma^{y}_{ay} = A^{-1}\partial_{a}A$. Using this relation (2.5) and the fact that $R_{\mu\nu\alpha\beta} = {}^{(5)}R_{\mu\nu\alpha\beta} + 2K_{\mu}{}_{[\alpha}K_{\beta]\nu}$, the five-dimensional Ricci scalar in (2.2) can be decomposed in terms of four-dimensional quantities [103]:

$${}^{(5)}R = {}^{(5)}R_{\alpha\mu\beta\nu}q^{\alpha\beta}q^{\mu\nu} + 2{}^{(5)}R^{y}_{\ \mu\nu\nu}q^{\mu\nu} = R - (K^2 - K_{\mu\nu}K^{\mu\nu}) - 2\mathcal{L}_nK - 2K_{\mu\nu}K^{\mu\nu} - 2A^{-1}\Box A, \qquad (2.6)$$

where $\Box = D^{\alpha}D_{\alpha}$. Substituting this expression for ⁽⁵⁾R in the action, the normal derivative of K can be integrated by parts. The surface term cancels with the Einstein-Hilbert boundary term K in (2.2) giving rise to the action:

$$S = \int d^5 x A \sqrt{-q} \left(\frac{1}{2\kappa} \left(R + K^2 - K_{\mu\nu} K^{\mu\nu} - 2A^{-1} \Box A \right) - \Lambda \right) + \int d^4 x \sqrt{-q} \mathcal{L}.$$
 (2.7)

Formally, S is a functional of the five-dimensional metric g^{ab} . Extremising S with respect to g^{ab} gives rise to the five-dimensional Einstein equation ${}^{(5)}G_{ab} = \kappa \Lambda g_{ab}$ in the bulk. Instead, we may consider S as being a function of A, $q^{\mu\nu}$ and n^{μ} and extremise $S[q^{\mu\nu}, n^{\mu}, A]$ with respect to these variables. Expressing $K_{\mu\nu}$ in terms of n^a and $q_{\mu\nu}$, using (2.3), we may extremise S with respect to n^{μ} . This gives rise to the Codacci equation [103, 104]:

$$D_{\alpha}K - D_{\mu}K^{\mu}_{\alpha} = 0. \tag{2.8}$$

The differentiation of the action with respect to $q^{\mu\nu}$ gives rise to

$$G_{\mu\nu}(y) = -\kappa \Lambda q_{\mu\nu} - \frac{1}{2}K^2 q_{\mu\nu} + 3KK_{\mu\nu} - \frac{1}{2}K_{\alpha\beta}K^{\alpha\beta}q_{\mu\nu} - 2K_{\mu\alpha}K^{\alpha}_{\nu} \qquad (2.9)$$
$$-A^{-1} \left(\Box A q_{\mu\nu} + D_{\mu}D_{\nu}A\right) - \mathcal{L}_n \left[Kq_{\mu\nu} - K_{\mu\nu}\right] + \frac{\kappa}{A} T^{\text{tot}}_{\mu\nu} \,\delta(y).$$

We may integrate the previous expression around \mathcal{M} and obtain the jump condition, better known as the Israël matching condition [105] which relates the jump of the extrinsic curvature across the hypersurface y = 0 to the stress-energy tensor $T_{\mu\nu}^{\text{tot}} = -\frac{\delta \sqrt{-g} \mathscr{L}}{2\sqrt{-g} \delta q^{\mu\nu}}$ on \mathcal{M} [102–104]:

$$\Delta \left[Kq_{\mu\nu} - K_{\mu\nu} \right] = \lim_{\epsilon \to 0} \int_{-\epsilon}^{+\epsilon} \mathrm{d}y \left(Kq_{\mu\nu} - K_{\mu\nu} \right) = \kappa T_{\mu\nu}^{\mathrm{tot}}.$$
 (2.10)

Decomposing $\mathcal{L}_n[Kq_{\mu\nu} - K_{\mu\nu}]$ into a continuous part and a delta function:

$$\mathcal{L}_{n} [Kq_{\mu\nu} - K_{\mu\nu}] = \frac{1}{A} [Kq_{\mu\nu} - K_{\mu\nu}]_{0^{-}}^{0^{+}} \delta(y)$$

$$+ \frac{3}{L^{2}}q_{\mu\nu} + 2KK_{\mu\nu} - K_{\mu\alpha}K_{\nu}^{\alpha} - K_{\alpha\beta}K^{\alpha\beta}q_{\mu\nu}$$

$$-A^{-1} (\Box A q_{\mu\nu} + D_{\mu}D_{\nu}A) + E_{\mu\nu},$$
(2.11)

the continuous part of eq.(2.9) can then be interpreted as the Gauss equation [102–104]:

$$R_{\mu\nu} = {}^{(5)}R_{\mu\nu} + (KK_{\mu\nu} - K_{\alpha\mu}K_{\nu}^{\alpha}) - E_{\mu\nu}$$

$$= -\frac{3}{L^2}q_{\mu\nu} + (KK_{\mu\nu} - K_{\alpha\mu}K_{\nu}^{\alpha}) - E_{\mu\nu},$$
(2.12)

which is the analogous of the Einstein equation on the hypersurface \mathcal{M} .

Finally, the differentiation of S with respect to A confirms the fact that $E_{\mu\nu}$ is traceless:

$$R = -\frac{12}{L^2} - K^2 + K_{\mu\nu}K^{\mu\nu}.$$
(2.13)

2.3.2 Gauss-Codacci Equations on the Boundary Branes

In the previous subsection, we derived the Gauss-Codacci equations for an arbitrary hypersurface \mathcal{M} located at y = 0. We now identify \mathcal{M} with the boundary branes of the RS scenario. These branes have the important property to be \mathbb{Z}_2 symmetric so the extrinsic curvatures $K_{\mu\nu}$ on each side of the branes are equal and opposite and can be uniquely determined using the Israël matching condition (2.10)

$$\Delta K_{\mu\nu} = \pm 2K_{\mu\nu}^{(\pm)} = -\kappa \left(T_{\mu\nu}^{(\pm)\,\text{tot}} - \frac{1}{3} T^{(\pm)\,\text{tot}} q_{\mu\nu}^{(\pm)} \right), \qquad (2.14)$$

where the \pm superscript indicates the induced value of the given quantity on the positive or negative tension brane. We now suppose that the branes have a tension λ^{\pm} which is fine-tuned to its canonical value and absorb any deviation from it to the stress-energy tensor for matter fields living on the branes $T^{(\pm)}_{\mu\nu}$

$$T_{\mu\nu}^{(\pm)\,\text{tot}} = -\lambda^{\pm} q_{\mu\nu}^{(\pm)} + T_{\mu\nu}^{(\pm)}$$
(2.15)

$$\lambda^{\pm} = \pm \frac{6}{\kappa L}.$$
(2.16)

The extrinsic curvature on each brane is therefore:

$$K^{\mu\,(\pm)}_{\nu} = -\frac{1}{L}\,\delta^{\mu}_{\nu} \mp \frac{\kappa}{2}\,\left(T^{\mu\,(\pm)}_{\nu} - \frac{1}{3}T^{(\pm)}\delta^{\mu}_{\nu}\right).$$
(2.17)

Using this expression in the Codacci condition (2.8), we recover the equation for the conservation of energy on the branes

$$D^{(\pm)}_{\mu}T^{\mu\,(\pm)}_{\nu} = 0. \tag{2.18}$$

Writing the extrinsic curvature in terms of the stress-energy tensor, and using the Gauss equation (2.12), the projected Ricci tensor on each brane can be expressed as [103, 104]

$$R_{\mu\nu}^{(\pm)} = \pm \frac{\kappa}{L} \left(T_{\mu\nu}^{(\pm)} - \frac{1}{2} T^{(\pm)} g_{\mu\nu}^{(\pm)} \right) - \frac{\kappa^2}{4} \left(T_{\mu}^{(\pm)\alpha} T_{\alpha\nu}^{(\pm)} - \frac{1}{3} T^{(\pm)} T_{\mu\nu}^{(\pm)} \right) - E_{\mu\nu}^{(\pm)}, \qquad (2.19)$$

where the fine-tuned value of the tensions λ^{\pm} is fixed to exactly cancel the contribution from ⁽⁵⁾ $R_{\mu\nu}$ in (2.12). We can remark on two features in this modified Einstein equation. The first one is the presence of terms quadratic in the stress-energy tensor. The aim of part of chapter 3 will be to understand the implications of those terms to braneworld cosmology. The second departure from the standard four-dimensional Einstein equation arises from the presence of the Weyl tensor $E_{\mu\nu}$ which is undetermined on the brane. It is worth pointing out that when the cosmological constant in the bulk is important, the length scale L is negligible compared to any other length scale present in the theory. In that case, the first term in (2.19) dominates and we recover standard four-dimensional gravity for the positive tension brane (up to the redefinition of the four-dimensional Planck mass $M_4^2 \equiv \frac{L}{\kappa}$).

2.4 Background Behaviour

We assume the five-dimensional Universe to be homogeneous and isotropic in the three spatial directions, therefore we consider both the bulk metric and the branes stress-energy tensors $T^{(\pm)}_{\mu\nu}$ to have the required cosmological symmetry. We may work in the frame where the bulk is static (the existence of such a frame is a consequence of Birkhoff's theorem [106], [100,107]). In that frame, the most general geometry is Schwarzschild-Antide-Sitter (SAdS) with parameter C associated with the Black Hole mass [98–100, 108]:

$$ds^{2} = dY^{2} - n^{2}(Y)dT^{2} + a^{2}(Y)d\Omega_{3}^{2}$$
(2.20)
with $a^{2}(Y) = e^{-2Y/L} + \frac{C}{4}e^{2Y/L} - \frac{kL^{2}}{2}$
 $n^{2}(Y) = L^{2}a'(Y)^{2} = a^{2} - \frac{C}{a^{2}},$

where k is the spatial curvature, and can take values -1, 0 or 1 depending if $d\Omega_3^2 = \mathbb{H}^3, \mathbb{R}^3$ or \mathbb{S}^3 . We may indeed check that (2.20) satisfies the five-dimensional Einstein equation $G_{ab} = \kappa \Lambda g_{ab}$.

In this frame, the branes are not static but have location $Y = Y_{\pm}(T)$. The Israël matching conditions, associated with the \mathbb{Z}_2 -symmetry, impose the condition on the extrinsic curvature (2.17), with the branes stress-energies:

$$T^{(\pm)\,0}_{\ \ 0} = -\rho_{\pm}, \qquad T^{(\pm)\,i}_{\ \ j} = p_{\pm}\,\delta^{i}_{j}. \tag{2.21}$$

On the brane, the spatial components of the extrinsic curvature are:

$$K_{j}^{i}(Y_{\pm}) = \left(1 - \frac{\dot{Y}_{\pm}^{2}}{n_{\pm}^{2}}\right)^{-1/2} \left. \frac{a'(Y)}{a(Y)} \right|_{Y=Y\pm} \delta_{j}^{i}, \qquad (2.22)$$

so using (2.17) the branes' velocities must satisfy:

$$\dot{Y}_{\pm}^{2} = \left(\frac{\mathrm{d}Y_{\pm}}{\mathrm{d}T}\right)^{2} = n_{\pm}^{2} \left(1 - \frac{n_{\pm}^{2}}{a_{\pm}^{2}} \left(1 \pm \frac{\kappa L}{6} \rho_{\pm}\right)^{-2}\right), \qquad (2.23)$$

where $a_{\pm}(T) \equiv a(Y = Y_{\pm}(T))$ is the value of the scale factor on each brane, and similarly for $n_{\pm}(T)$. The induced line element on the branes can be read off as

$$ds_{\pm}^{2} = -(n_{\pm}^{2} - \dot{Y}_{\pm}^{2})dT^{2} + a_{\pm}^{2}d\mathbf{x}^{2}$$

$$\equiv a_{\pm}^{2} \left(-d\tau_{\pm}^{2} + d\mathbf{x}^{2}\right), \qquad (2.24)$$

defining the conformal time on each brane:

$$\frac{\mathrm{d}\tau_{\pm}}{\mathrm{d}T} \equiv \sqrt{\frac{n_{\pm}^2 - \dot{Y}_{\pm}^2}{a_{\pm}^2}} = \frac{n_{\pm}^2}{a_{\pm}^2} \left(1 \pm \frac{\kappa L}{6} \rho_{\pm}\right)^{-2}.$$
(2.25)

The evolution of the scale factor on each brane in terms of their respective conformal time is:

$$\left(\frac{\mathrm{d}a_{\pm}}{\mathrm{d}\tau_{\pm}}\right)^{2} = \dot{Y}_{\pm}^{2} \left(\frac{\mathrm{d}T}{\mathrm{d}\tau_{\pm}}\right)^{2} \left.\frac{\mathrm{d}a(Y)}{\mathrm{d}Y}\right|_{Y=Y_{\pm}}^{2} = \frac{a_{\pm}^{4}}{L^{2}} \left(\left(1 \pm \frac{\kappa L}{6}\rho_{\pm}\right)^{2} - \frac{n_{\pm}^{2}}{a_{\pm}^{2}}\right).$$
(2.26)

It follows that each of the two branes satisfies the induced or modified Friedmann equation:

$$H_{\pm}^{2} = \frac{1}{L^{2}} \left(\left(1 \pm \frac{\kappa L}{6} \rho_{\pm} \right)^{2} - \frac{n_{\pm}^{2}}{a_{\pm}^{2}} \right)$$
(2.27)

$$= -\frac{k}{a_{\pm}^2} \pm \frac{\kappa}{3L} \rho_{\pm} + \frac{\kappa^2}{36} \rho_{\pm}^2 + \frac{\dot{C}}{L^2 a_{\pm}^4}, \qquad (2.28)$$

where $\tilde{C} = C - \frac{k^2 L^4}{4}$. This is a very well-known result pointed out in a large number of work [98–100, 103, 104, 108–111]. The same features as in the modified Einstein equation (2.19) can be pointed out, namely the presence of the quadratic term in the energy density as well as the presence of an extra-term $\frac{\tilde{C}}{L^2 a_{\pm}^4}$ looking like radiation. This last term can be associated to the contribution from the Weyl tensor in (2.19). Indeed, knowing the bulk geometry exactly, the Weyl tensor $E_{\mu\nu}$ defined in (2.4) can be calculated. For homogeneous and isotropic metrics, the value of $E_{\mu\nu}$ on the branes can be seen to have the form of the stress-energy tensor for radiation:

$$E^{(\pm)\,0}_{\ \ 0} = -\frac{3\,\hat{\mathcal{C}}}{L^2 a_{\pm}^4}, \quad E^{(\pm)\,i}_{\ \ j} = \frac{\hat{\mathcal{C}}}{L^2 a_{\pm}^4} \delta^i_j. \tag{2.29}$$

These are the two main features that distinguish braneworld cosmology to any ordinary four-dimensional cosmological scenario. This is the reason why this thesis will have for main focus to study the effects of these terms in a general cosmological setup. For this, it is essential to have a closed set of covariant equation that describes the brane geometry. The modified Einstein equation (2.19) is a good start, but it contains information from the Weyl tensor $E_{\mu\nu}$ which is a priori undefined on the brane. The purpose of this thesis will be to derive as much information as possible for the behaviour of $E_{\mu\nu}$ in different scenarios. We shall therefore start in the next subsection by deriving its equation of motion in the bulk.

2.5 Evolution of the Weyl Tensor

On the brane, the only quantity which remains to be defined is the electric part of the Weyl tensor $E_{\mu\nu}$. In this subsection we shall find the evolution equation of the Weyl tensor in the bulk in order to have a closed set of equations for the induced geometry on the branes.

We consider the frame (2.1) for which both branes are static and located at y = 0 (for the positive tension brane) and y = 1 (for the negative one). We recall that \mathcal{L}_n is the Lie derivative with respect to the normal vector of a y = const hypersurface and for any tensor lying purely in this hypersurface, the action of \mathcal{L}_n is equivalent to $A^{-1}\partial_y$.

In the frame (2.1), the expression for the Weyl tensor is derived in (2.5). In particular, we can see that $E_{\mu\nu}$ is associated with the normal derivative of the extrinsic curvature which itself is the normal derivative of the induced metric $q_{\mu\nu}$ on a y = const hypersurface. Substituting this expression for $E_{\mu\nu}$ into the Gauss equation (2.12), we therefore have a non-linear second order differential equation for $q_{\mu\nu}$. This equation however is extremely difficult to solve (even for cosmological symmetry). Another way to proceed is instead to differentiate (2.12) recursively as we shall see in chapter 5. For this we shall need the Lie derivative of $E_{\mu\nu}$ [112].

We denote by $\Gamma^{\alpha}_{\mu\nu}$ the four-dimensional connection associated with $q_{\mu\nu}$. Its Lie derivative is:

$$\mathcal{L}_{n}\Gamma^{\alpha}_{\mu\nu} = \frac{1}{A} \Big(D_{\nu} \left(AK^{\alpha}_{\mu} \right) + D_{\mu} \left(AK^{\alpha}_{\nu} \right) - D^{\alpha} \left(AK_{\mu\nu} \right) \Big), \tag{2.30}$$

giving rise to the evolution of the four-dimensional Riemann tensor:

$$\mathcal{L}_{n} R^{\mu}_{\ \nu\alpha\beta} = \frac{2}{A} \Big(D_{[\alpha} D_{\nu} (AK^{\mu}_{\ \beta]}) + D_{[\alpha} D_{\beta]} (AK^{\mu}_{\ \nu}) - D_{[\alpha} D^{\mu} (AK_{\beta]\nu}) \Big), \tag{2.31}$$

and the evolution of the Ricci tensor:

$$\mathcal{L}_n R_{\beta\nu} = \frac{1}{A} \left(D_\alpha D_\nu (AK^\alpha_\beta) + D_\alpha D_\beta (AK^\alpha_\nu) - \Box (dK_{\beta\nu}) - D_\nu D_\beta (AK) \right).$$
(2.32)

Using this relation into the Gauss equation (2.12), together with (2.5) and the relation

 $\mathcal{L}_n R^{\mu}_{\nu} = q^{\mu\beta} \mathcal{L}_n R_{\beta\nu} - 2K^{\mu\beta} R_{\beta\nu}$, we therefore have [113]

$$\mathcal{L}_{n}E_{\nu}^{\mu} = \mathcal{L}_{n}R_{\nu}^{\mu} + \mathcal{L}_{n}\left(K_{\alpha}^{\mu}K_{\nu}^{\alpha} - KK_{\nu}^{\mu}\right) = 2K_{\nu}^{\alpha}E_{\alpha}^{\mu} - \frac{3}{2}KE_{\nu}^{\mu} - \frac{1}{2}K_{\beta}^{\alpha}E_{\alpha}^{\beta}\delta_{\nu}^{\mu} + C_{\alpha\nu\beta}^{\mu}K^{\alpha\beta} + (K^{3})_{\nu}^{\mu} \qquad (2.33) - \frac{1}{2A^{2}}D^{\alpha}\left[A^{2}D^{\mu}K_{\alpha\nu} + A^{2}D_{\nu}K_{\alpha}^{\mu} - 2A^{2}D_{\alpha}K_{\nu}^{\mu}\right],$$

where $(K^3)^{\mu}_{\nu}$ are some cubic terms in the traceless part of the extrinsic curvature whose exact form will not be relevant for the purpose of this study (it is enough to note that this term vanishes if $K^{\mu}_{\nu} \sim \delta^{\mu}_{\nu}$). This set of equations is still very difficult to study, and unless some special limits are considered very little analytical progress can be made. At low energies however, the geometry on the branes simplifies remarkably and is very close to standard four-dimensional theories since the bulk degrees of freedom are not excited.

2.6 Low-energy Effective Theory and Four-dimensional Action

2.6.1 Low-energy Regime

As we shall see below, at low energies the system is well-described by a four-dimensional effective theory. In this limit, braneworlds behave like conventional scalar-tensor theory of gravity where the bulk effects are represented by a single scalar field.

Many different derivations of the low-energy effective theory can be found in the literature [96, 111, 113, 114]. Among them, the most intuitive derivation is probably the one arising from the moduli space approximation (first defined in [115] and applied to braneworld models in [86, 116–119]), but we choose to derive it in a more covariant way and remain in the frame (2.1) where the branes are static. As already mentioned, when the bulk cosmological constant is important, the bulk warping strongly localises gravity on the very massive brane and we expect to recover standard four-dimensional gravity for the positive tension brane. This is the limit we study here, in particular we assume the AdS
length scale L to be much smaller than any other length scale l along the four transverse directions, and L^{-1} much greater than any derivative scale. This is therefore equivalent to neglecting any derivatives higher than second order in the theory. The resulting theory should therefore only be valid at long wavelengths and at small velocities. However this does not mean that the distance d between the branes is large. The distance between the branes is not associated with the length scale l along the four transverse directions and taking the low-energy limit is not equivalent to taking the limit $L \ll d$. This theory is hence valid for branes about to collide as much as for the one-brane limit of the model, provided that the system is at low-energy $L \ll l$.

Using the low-energy limit, we will expand the extrinsic curvature $K_{\mu\nu}$ and the Weyl tensor $E_{\mu\nu}$ as power in derivatives.

2.6.2 Leading Order

To lowest order in derivatives, the equations (2.5) and (2.12) are simply:

$$-\frac{3}{L^2}\delta^{\mu}_{\nu} + \left(K^{(0)}K^{(0)\mu}_{\ \nu} - K^{(0)\mu}_{\ \alpha}K^{(0)\alpha}_{\ \nu}\right) - E^{(0)\mu}_{\ \nu} = A^{-1}\partial_y K^{(0)\mu}_{\ \nu}$$
(2.34)

$$-E^{(0)\,\mu}_{\ \nu} - K^{(0)\,\mu}_{\ \alpha} K^{(0)\,\alpha}_{\ \nu} + \frac{1}{L^2} \delta^{\mu}_{\nu} = 0, \qquad (2.35)$$

which imposes $E^{(0)}{}^{\mu}{}_{\nu}$ to vanish and $K^{(0)}K^{(0)}{}^{\mu}{}_{\nu} = \frac{4}{L^2}\delta^{\mu}_{\nu}$, such that $K^{(0)}{}^{\mu}{}_{\nu} = -\frac{1}{L}\delta^{\mu}_{\nu}$, (the solution $K^{(0)}{}^{\mu}{}_{\nu} = \frac{1}{L}\delta^{\mu}_{\nu}$ is not consistent with the boundary conditions (2.17)). We may check that this solution is consistent with the evolution equation for $E_{\mu\nu}$ (2.33) to zeroth order in derivatives. Recalling from (2.3) that the extrinsic curvature is $K_{\mu\nu} = A^{-1}\partial_y q_{\mu\nu}$, to leading order in L, the induced metric on a y = const hypersurface is therefore [113]:

$$q_{\mu\nu}^{(0)}(y,x) = e^{-2\tilde{d}(y,x)/L} q_{\mu\nu}^{(+)}(x), \qquad (2.36)$$

where $q_{\mu\nu}^{(+)}(x)$ is the induced metric on the positive tension brane and $\tilde{d}(y,x)$ is the proper distance between the positive tension brane and the hypersurface located at y = const: $\tilde{d}(y,x) = \int_0^y A(y',x) dy'$. This is a remarkable result: At low-energy, the induced metrics on y = const hypersurfaces are all conformal to each other [120], and in particular the positive and negative tension branes are related by the conformal factor Ψ

$$q_{\mu\nu}^{(-)}(x) = \Psi^2 q_{\mu\nu}^{(+)}(x)$$
(2.37)

$$\Psi = e^{-d/L}, \qquad (2.38)$$

where d(x) is the proper distance between the branes: $d(x) = \int_0^1 A \, dy$. We note that this result has been established without using any assumption on the background geometry, in particular, this result is still valid even if the background geometry is not AdS.

Since to leading order, the metric on both branes are conformally related with factor $\Psi = e^{-d/L}$, the induced Einstein tensor on each brane must satisfy, at low-energy:

$$G_{\mu\nu}^{(+)} = G_{\mu\nu}^{(-)} - \frac{2}{L} \left(D_{\mu}^{(+)} D_{\nu}^{(+)} d - \Box^{(+)} d q_{\mu\nu}^{(+)} + \frac{1}{L} \partial_{\mu} d \partial_{\nu} d + \frac{1}{2L} \left(\partial d \right)^2 q_{\mu\nu}^{(+)} \right).$$
(2.39)

2.6.3 Expression for the Weyl Tensor

To first order in derivatives, using the previous result, the evolution equation for $E_{\mu\nu}$ (2.33) implies (using the fact that $E_{\mu\nu}$ is traceless)

$$A^{-1}\partial_{y}E^{(1)}{}^{\mu}_{\nu} = \frac{4}{L}E^{(1)}{}^{\mu}_{\nu}, \qquad (2.40)$$

whose solution is simply:

$$E^{(1)\mu}_{\ \nu}(y,x) = e^{4\tilde{d}(y,x)/L} E^{(+)\mu}_{\ \nu}(x).$$
(2.41)

If we stop the expansion at this order, the Weyl tensors on each brane are as well conformally related:

$$\Psi^2 E^{(-)}_{\mu\nu} = E^{(+)}_{\mu\nu}, \qquad (2.42)$$

but using (2.19), the induced Einstein tensor are

$$G_{\mu\nu}^{(\pm)} = -E_{\mu\nu}^{(\pm)} \pm \frac{\kappa}{L} T_{\mu\nu}^{(\pm)}, \qquad (2.43)$$

(the quadratic terms in the stress-energy tensor are negligible at low-energy). This implies the second relation between the induced Einstein tensor on each brane:

$$G_{\mu\nu}^{(+)} - \Psi^2 G_{\mu\nu}^{(-)} = \frac{\kappa}{L} \left(T_{\mu\nu}^{(+)} + \Psi^2 T_{\mu\nu}^{(-)} \right).$$
(2.44)

Using (2.44) in (2.39) we therefore have the modified Einstein equation on each brane

$$G_{\mu\nu}^{(\pm)} = \frac{2}{L} \frac{e^{-2(d-\bar{y}_{\pm})/L}}{1-\Psi^2} \left(D_{\mu}^{(\pm)} D_{\nu}^{(\pm)} d - \Box^{(\pm)} d q_{\mu\nu}^{(\pm)} \pm \frac{1}{L} \partial_{\mu} d \partial_{\nu} d \pm \frac{1}{2L} \left(\partial d \right)^2 q_{\mu\nu}^{(\pm)} \right) (2.45) + \frac{1}{1-\Psi^2} \frac{\kappa}{L} \left(T_{\mu\nu}^{(+)} + \Psi^2 T_{\mu\nu}^{(-)} \right),$$

where $\bar{y}_{+} = 0$ and $\bar{y}_{-} = d$. Comparing (2.45) with (2.43), at low energies, the expression for the Weyl tensor is therefore:

$$E_{\mu\nu}^{(\pm)} = -\frac{2}{L} \frac{e^{-2(d-\bar{y}_{\pm})/L}}{1-\Psi^2} \left[D_{\mu}^{(\pm)} D_{\nu}^{(\pm)} d - \Box^{(\pm)} d q_{\mu\nu}^{(\pm)} \pm \frac{1}{L} \partial_{\mu} d \partial_{\nu} d \pm \frac{1}{2L} \left(\partial d \right)^2 q_{\mu\nu}^{(\pm)} \right. (2.46) \\ \left. + \frac{\kappa}{2} \left(T_{\mu\nu}^{(+)} + T_{\mu\nu}^{(-)} \right) \right].$$

Furthermore, the tracelessness of the Weyl tensor imposes the equation of motion for d:

$$R^{(\pm)} = \mp \frac{\kappa}{L} T^{\pm}$$
(2.47)

$$\Box^{(+)}d = \frac{(\partial d)^2}{L} + \frac{\kappa}{6} \left(T^{(+)} + \Psi^2 T^{(-)}\right)$$
(2.48)

$$\Box^{(-)}d = -\frac{(\partial d)^2}{L} + \frac{\kappa}{6} \left(T^{(-)} + \Psi^{-2}T^{(+)}\right).$$
(2.49)

At low-energy, it is therefore possible to find an effective four-dimensional theory on the branes. The resulting theory is simply gravity conformally coupled to a scalar field Ψ (or d). This theory strongly relies on the fact that both the brane metrics and the induced Weyl tensor are conformally related on each brane. At higher-order in derivatives, we do not expect this to be true (again this is true, independently from the brane separation). As we shall see in section (2.7), this effective low-energy theory actually models the zero mode of the tower of Kaluza Klein excitations *ie*. this effective theory is correct as long as no bulk degrees of freedom (or Kaluza Klein modes) are excited, this is as expected for a low-energy theory. Since this theory looks like a conventional scalar-tensor theory of gravity, it might be possible to find a frame for which gravity is minimally coupled to the scalar field and to derive it from an action. But first we may check the consistency of this effective theory for the background, where it is possible to compare its predictions with the five-dimensional results of section 2.4.

2.6.4 Background Behaviour

Assuming cosmological symmetry, we can find the Friedmann equation on each brane. Its expression is simply a consequence of (2.47) which imposes the Ricci tensor on each brane to be $R^{(\pm)} = 6\left(\frac{k^2}{a_{\pm}^2} + \frac{\dot{a}_{\pm}^2}{a_{\pm}^2} + \frac{\ddot{a}_{\pm}}{a_{\pm}}\right) = \pm \frac{\kappa}{L} (\rho_{\pm} - 3p_{\pm})$, where a "dot" designates a derivative with respect to the proper time on the brane. Using this relation together with the conservation of energy $\dot{\rho}_{\pm} = -3\frac{\dot{a}_{\pm}}{a_{\pm}} (\rho_{\pm} + p_{\pm})$, the Friedmann equation on the branes is simply

$$H_{\pm}^{2} = -\frac{k^{2}}{a_{\pm}^{2}} \pm \frac{\kappa}{3L}\rho_{\pm} + \frac{\tilde{D}_{\pm}}{L^{2}a_{\pm}^{4}},$$
(2.50)

where \tilde{D}_{\pm} is an integration constant. This Friedmann equation looks very similar to the exact result for the background (2.28) at low-energy, when $\rho_{\pm} \ll 1/\kappa L$ and the quadratic terms in the energy density may be neglected. It is important to point out that this is only a consequence of the tracelessness of the Weyl tensor. The four-dimensional low-energy theory models the background correctly to that level only because it reproduces the fact that the Weyl tensor $E_{\mu\nu}$ is traceless. The exact expression for the integration constant \tilde{D}_{\pm} has to be derived from the modified Einstein equation (2.45), and in particular there is no reason why it should reproduce the exact result obtain from the five-dimensional theory, we shall indeed see in chapter 5 that this is actually not the case.

2.6.5 Einstein Frame

So far, the low-energy effective theory has been derived in the brane frame. In the modified Einstein equation (2.45), the presence of the terms of the form D^2d implies that if this Einstein equation was derived from an action, this action should include a term of the form $f(d)R^{(\pm)}$. In order to derive these equations from an action describing a scalar field minimally coupled to gravity, we may perform the change of frame:

$$q_{\mu\nu}^{(\pm)} = \Omega_{\pm}^2 \bar{g}_{\mu\nu}, \tag{2.51}$$

$$\Omega_{+} = \cosh\left(\phi/\sqrt{6}\right), \quad \Omega_{-} = -\sinh\left(\phi/\sqrt{6}\right),$$
(2.52)

where in what follows, any "bar" quantity will designate a quantity with respect to this new frame, which is the Einstein frame as we shall see below. Since the conformal factor relating the two branes is Ψ , we have the relation [117]:

$$\Psi = e^{-d/L} = -\tanh\left(\phi/\sqrt{6}\right). \tag{2.53}$$

In this frame, the Einstein equation (2.45) simplifies to

$$\bar{G}_{\mu\nu} = \partial_{\mu}\phi \,\partial_{\nu}\phi - \frac{1}{2} \left(\partial d\right)^2 \bar{g}_{\mu\nu} + \frac{\kappa}{L} \left(\Omega_+^2 T_{\mu\nu}^{(+)} + \Omega_-^2 T_{\mu\nu}^{(-)}\right), \qquad (2.54)$$

with the equation for the scalar field:

$$\bar{\Box}\phi = -\frac{\kappa}{L}\Omega^4_{+,\phi}\mathscr{L}_+ - \frac{\kappa}{L}\Omega^4_{-,\phi}\mathscr{L}_-.$$
(2.55)

This low-energy effective theory can therefore be derived from the effective action of a scalar field minimally coupled to gravity with non-minimally coupled matter [116]:

$$S = \frac{L}{\kappa} \int d^4x \sqrt{-\bar{g}} \left(\frac{1}{2} \bar{R} - \frac{1}{2} \left(\partial \phi \right)^2 \right) + S_m^-[g^-] + S_m^+[g^+], \qquad (2.56)$$

where $S_m^{\pm}[g^{\pm}] = \int dx^4 \sqrt{-g_{\pm}} \mathscr{L}_{\pm} = \int dx^4 \sqrt{-\bar{g}} \Omega_{\pm}^4 \mathscr{L}_{\pm}$ are the conventional four-dimensional matter actions on each brane.

In the one-brane limit, when the distance between the branes is taken to infinity, $\phi \to 0$, the positive tension brane is equivalent to the Einstein frame $\Omega_+ \to 1$ and we recover standard gravity minimally coupled to a scalar field on that brane. Its coupling to matter on the negative tension brane indeed vanishes since $\Omega_- \to 0$. However, from the negative brane point of view, gravity becomes highly coupled to the matter on the positive tension brane.

2.7 Kaluza Klein modes for Static Branes

2.7.1 Derivative Expansion

For the purpose of this study we will consider the branes to be spatially flat and static for the background. Using the Israël matching condition (2.23), this is possible if the background geometry is pure AdS (no bulk Black Hole, C = 0) and if the branes are empty $T^{(\pm)}_{\mu\nu} = 0$ in the background. To study this system, it is easier to work in the frame of the form:

$$ds^{2} = dy^{2} + e^{-2|y|/L} \eta_{\mu\nu} dx^{\mu} dx^{\nu}, \qquad (2.57)$$

with the branes located at y = 0 and y = d (for the background), and their induced metric is $q_{\mu\nu}^{(+)} = \eta_{\mu\nu}$, $q_{\mu\nu}^{(-)} = e^{-2d/L}\eta_{\mu\nu}$. We shall denote by $\Psi_0 = e^{-d/L}$ the constant conformal factor relating the background metrics on the two branes.

Following the procedure of [121], we consider the metric perturbations sourced by matter confined on the positive tension brane, with stress-energy tensor $\delta T^{(+)}_{\mu\nu}$. Since matter is introduced as a perturbation, to first order, the matter fields confined on the brane are living on the background flat metric $q^{(+)}_{\mu\nu}$.

As considered in [121], we shall first solve the equations of motion for the perturbed metric in RS gauge. In this gauge, the position of the positive tension brane is not fixed. We denote by δy the deviation from its fixed background location. In order to express the perturbed metric induced on each brane, we will later change to Gaussian normal coordinates. When the distance between the two branes is finite, the perturbed metric on the branes can then be expanded in momentum space.

In RS gauge, the perturbed metric is:

$$ds^{2} = dy^{2} + \left(e^{-2|y|/L}\eta_{\mu\nu} + h_{\mu\nu}\right)dx^{\mu}dx^{\nu}$$
(2.58)

with
$$h^{\mu}_{\mu} = 0$$
 and $h^{\mu}_{\nu,\mu} = 0$, (2.59)

where $h_{\mu\nu}$ is transverse and traceless with respect to the background metric $q_{\mu\nu} = e^{-2|y|/L} \eta_{\mu\nu}$. In this gauge, the five-dimensional Einstein equation and the Israël matching conditions read:

$$\left[e^{2y/L} \Box + \partial_y^2 - \frac{4}{L^2} \right] h_{\mu\nu} = 0 \text{ for } 0 < y < d, \qquad (2.60)$$

$$\left[\partial_y + \frac{2}{L}\right] h_{\mu\nu} = \begin{cases} -\kappa \Sigma_{\mu\nu} & \text{at } y = 0\\ 0 & \text{at } y = d \end{cases},$$
(2.61)

where \Box is the Laplacian on the four-dimensional Minkowski space. The tensor $\Sigma_{\mu\nu}$ is a functional of the matter on the positive brane: $\Sigma_{\mu\nu} = \delta T_{\mu\nu}^{(+)} - \frac{1}{3}\delta T^{(+)}\eta_{\mu\nu} - \frac{2}{\kappa}\delta y_{,\mu\nu}$, with $\delta T^{(+)} = \eta^{\mu\nu} \delta T_{\mu\nu}^{(+)}$. From eqs.(2.61) and (2.59), the tensor $\Sigma_{\mu\nu}$ must be transverse and traceless with respect to its background flat metric $q_{\mu\nu}^{(+)} = \eta_{\mu\nu}$. The deviation δy from the background position is therefore given by $\delta y = -\frac{\kappa}{6} \frac{1}{\Box} \delta T^{(+)}$:

$$\Sigma_{\mu\nu} = \delta T^{(+)}_{\mu\nu} - \frac{1}{3} \delta T^{(+)} \eta_{\mu\nu} + \frac{1}{3\Box} \delta T^{(+)}_{,\mu\nu}.$$
(2.62)

The differential equation (2.60) with boundary conditions (2.61) can be solved, giving the expression for the perturbed metric in RS gauge:

$$h_{\mu\nu}(y,x^{\mu}) = \kappa \,\hat{F}(y)\Sigma_{\mu\nu},\tag{2.63}$$

where the operator \hat{F} can be expressed as:

$$\hat{F}(y) = \frac{1}{\sqrt{-\Box}} \left[\frac{I_2(e^{y/L}L\sqrt{-\Box})K_1(e^{d/L}L\sqrt{-\Box}) + I_1(e^{d/L}L\sqrt{-\Box})K_2(e^{y/L}L\sqrt{-\Box})}{I_1(e^{d/L}L\sqrt{-\Box})K_1(L\sqrt{-\Box}) - I_1(L\sqrt{-\Box})K_1(e^{d/L}L\sqrt{-\Box})} \right], (2.64)$$

with I_n (resp. K_n) the n^{th} Bessel function of first (resp. second) kind.

Since in RS gauge, the negative tension brane remains located at $y \equiv d$, $h_{\mu\nu}(x^{\mu}, y = d)$ is hence the metric perturbation induced on that brane. However, this is not the case for the positive tension brane. In order to find the induced metric on that brane, we need to perform a gauge transformation and work in terms of the Gaussian normal (GN⁽⁺⁾) gauge for this brane:

$$\begin{cases} \bar{y} = y - \delta y \\ \bar{x}^{\mu} = x^{\mu} + \zeta^{\mu}(x^{\nu}). \end{cases}$$
(2.65)

When working in the GN⁽⁺⁾ gauge, with coordinates (\bar{y}, \bar{x}^{μ}) , the positive tension brane is located at $\bar{y} \equiv 0$. Performing the gauge transformation (2.65) with $\delta y = -\frac{\kappa}{6} \frac{1}{\Box} \delta T^{(+)}$, the perturbed metric induced on the positive tension brane is given by:

$$\bar{h}^{+}_{\mu\nu}(\bar{y}\equiv 0) = h_{\mu\nu}(y=0) + \frac{\kappa}{3L}\eta_{\mu\nu}\frac{1}{\Box}\delta T^{(+)} - \zeta_{(\mu,\nu)}, \qquad (2.66)$$

where the indices are raised and lowered with the metric $\eta^{\mu\nu}$. We may fix the remaining degrees of freedom by imposing the gauge choice:

$$\bar{h}^{\mu}_{\nu,\mu} = \frac{1}{2} \bar{h}^{\mu}_{\mu,\nu} \text{ at } \bar{y} = 0.$$
(2.67)

This gauge choice corresponds to the de Donder gauge and is obtained by fixing $\zeta^{\mu}(x^{\nu})$ such that $\zeta_{\mu} = -\frac{\kappa}{3L\Box^2} \delta T^{(+)}_{,\mu}$.

In de Donder gauge, the metric perturbation induced on the positive tension brane is therefore:

$$\bar{h}_{\mu\nu}^{(+)} = h_{\mu\nu}(y=0) + \frac{\kappa}{3L}\eta_{\mu\nu}\frac{1}{\Box}\delta T^{(+)} + \frac{2\kappa}{3L}\frac{1}{\Box^2}\delta T^{(+)}_{,\mu\nu}.$$
(2.68)

This leads to the perturbed metric on both branes:

$$\bar{h}_{\mu\nu}^{(+)} = \kappa \left(\hat{F}(0) \Sigma_{\mu\nu} + \frac{1}{3L} \eta_{\mu\nu} \frac{1}{\Box} \delta T^{(+)} + \frac{2}{3L} \frac{1}{\Box^2} \delta T^{(+)}_{,\mu\nu} \right), \qquad (2.69)$$

$$\bar{h}^{(-)}_{\mu\nu} = \kappa \,\hat{F}(d)\Sigma_{\mu\nu}.\tag{2.70}$$

We now focus on the interesting fact that the expression (2.64) of \hat{F} has a derivative expansion:

$$\hat{F}(y) = \frac{2L e^{-2y/L}}{1 - e^{-2d/L}} \left[-\frac{1}{L^2 \Box} + f(y) + \mathcal{O}(L^2 \Box) \right],$$
(2.71)

$$f(y) = \frac{d/2L}{1 - e^{-2d/L}} - \frac{1}{8} \left(1 - e^{2(2y-d)/L} + 2e^{2y/L} \right).$$
(2.72)

The first term in (2.71) can be interpreted as the zero mode and the second one as the first Kaluza Klein (KK) mode from the infinite KK tower. In the limit where $d \to \infty$, the expansion in (2.71) is ill-defined as the function f in (2.72) diverges and dominates over the zero-mode. In the limit of one positive brane embedded in a non-compactified fifth dimension (in the second RS model), a derivative expansion is not possible. There is indeed no mass gap between each mode in the KK tower which becomes a continuum when the fifth dimension is infinite.

To first order in derivatives, the perturbed metrics induced on each brane in their respective de Donder gauge are therefore:

$$\bar{h}_{\mu\nu}^{(+)} = -\frac{2L\kappa}{1-\Psi_0^2} \left[\frac{1}{L^2\Box} - f(0) + \mathcal{O}(L^2\Box) \right] \Sigma_{\mu\nu} + \frac{\kappa}{3L} \eta_{\mu\nu} \frac{1}{\Box} \delta T^{(+)} + \frac{2\kappa}{3L} \frac{1}{\Box^2} \delta T^{(+)}_{,\mu\nu}, \quad (2.73)$$
$$\bar{h}_{\mu\nu}^{(-)} = -\frac{2L\kappa\Psi_0^2}{1-\Psi_0^2} \left[\frac{1}{L^2\Box} - f(d) + \mathcal{O}(L^2\Box) \right] \Sigma_{\mu\nu}. \quad (2.74)$$

From this expression, we can read off the first KK mode on the branes. It is worth pointing out that in the limit of small distance $d \ll L$, we may expand \hat{F} as a series in d instead of derivatives. In that expansion, we recover the presence of a zero mode and KK corrections

$$h_{\mu\nu}(x^{\mu},0) = \frac{2\kappa}{L(1-e^{-2d/L})} \left[-\frac{1}{\Box} + \frac{d^2}{3} + d^2 \mathcal{O}(d/L) \right] \Sigma_{\mu\nu}$$
(2.75)

$$h_{\mu\nu}(x^{\mu},d) = \frac{2\kappa e^{-2d/L}}{L(1-e^{-2d/L})} \left[-\frac{1}{\Box} - \frac{d^2}{6} + d^2 \mathcal{O}(d/L) \right] \Sigma_{\mu\nu}.$$
 (2.76)

This result will be useful for the following. In the small d limit, the mass gap between each KK modes is important, and it is hence natural to decompose the tower and study the different modes one after the other.

An important point to notice in the previous result is that beyond the zero mode, the brane metrics are no longer conformal to each other (as would be predicted from the four-dimensional low-energy theory). The zero mode is very well described by the lowenergy effective theory of section 2.6, and the conformal factor $\Psi^2 = e^{-2d/L}$ is precisely the one expected from (2.37). This comes with no surprise as the branes are static and empty for the background and the low-energy condition is therefore very well respected. Even though, beyond the zero mode, no KK corrections could be predicted by that theory. The low-energy effective theory neglects any contribution from the bulk degree of freedom, which is the reason why it appears as a standard four-dimensional theory.

2.7.2 Kaluza Klein Correction Outside a Source in Real Space

In the case of a single brane in a non-compact infinite extra dimension, it can be shown [51,104,121] that the KK modes add a correction of order $1/r^3$ to the gravitational potential around a source point. If we consider a spherically symmetric source such that the stress-energy tensor on the brane is $T_{\mu\nu} = \rho(r)u_{\mu}u_{\nu}$. The potential can therefore be expressed in terms of the Green function for the Laplacian operator as $V = \frac{\kappa}{2} \int d^3x' G(x, x')\rho(x')$. With the Green function on the brane:

$$G(x,x') = -\int dk \frac{k\sin(kr)}{2\pi^2 r} \left[\frac{1}{Lk^2} + \int_0^\infty \frac{2e^{-mr}dm}{Lm(m^2 + k^2)(J_1(mL)^2 + Y_1(mL)^2)} \right] (2.77)$$

$$\simeq -\frac{1}{4L\pi r} + \frac{1}{\pi^2} \int_0^\infty \left(\frac{L\pi m}{8r} e^{-mr} + \cdots \right) dm \qquad (2.78)$$

$$\simeq -\frac{1}{4L\pi r} \left(1 + \frac{L^2}{2r^2} + \cdots \right), \qquad (2.79)$$

with r = |x - x'|, giving rise to the r^{-3} modification at long wavelengths in the potential: $V = -\frac{\kappa \int d^3x' \rho(x')}{8L\pi r} \left(1 + \frac{L^2}{2r^2} + \cdots\right).$

However, in the case of a compactified extra-dimension it is not possible to express the corrections as an expansion in real space (this is to be expected since an expansion in derivatives (*ie.* in momentum-space) is possible). Considering the extra-dimension to be compactified in an S^1/\mathbb{Z}_2 -orbifold of dimension d, this provides a mass gap between the modes in the infinite KK tower, and the expression of the Green function for the Laplacian operator is modified to:

$$G(x, x') \simeq -\frac{1}{4L\pi r} + \frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{L\pi n}{8d^2 r} e^{-nr/d} + \cdots$$
 (2.80)

$$\simeq -\frac{1}{4L\pi r} \left(1 + \frac{L^2}{2d^2} \frac{e^{-r/d}}{\left(1 - e^{-r/d}\right)^2} + \cdots \right).$$
 (2.81)

In the limit where $d \to \infty$ it is possible to express the corrections as a power expansion in real space at long wavelengths, but as soon as the extra-dimension is finite, at long wavelengths, the potential looks like a Yukawa kind of potential mediated by a massive particle of mass m = 1/d, $V = -\frac{\kappa \int d^3x' \rho(x')}{8L\pi r} \left(1 + \frac{L^2}{2d^2} e^{-r/d} + \cdots\right)$ which is precisely the kind of corrections expected from Kaluza Klein theory.

2.8 Extensions

2.8.1 Bulk Scalar Field and Potential for the Radion

The RS scenario represents a simple model for which no bulk scalar fields are present and the branes tension are fine-tuned to their canonical value. Although this model is rich in new physics, it remains a toy model for which no stabilisation mechanism is possible. A natural supersymmetric extension of this model is a BPS braneworld setup for which scalar fields Φ_I with potential $\mathscr{V}(\Phi_I)$ are present in the five-dimensional bulk. The potential \mathscr{V} replaces the rôle of the cosmological constant and the branes tension are assumed to be functions of the bulk scalar fields $\lambda_{\pm} = \lambda_{\pm} (\Phi_I(y_{\pm}))$. λ_{\pm} and $\mathscr{V}(\Phi_I(y_{\pm}))$ are related by a

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BPS condition when they are such that the effective four-dimensional cosmological constant on the branes cancels, this is equivalent to the fine-tuned condition for the brane tension in the RS model. For this supersymmetric configurations, the position of the brane is arbitrary as in the RS model and therefore the radion is a moduli. Instead one can consider a scenario for which the relation between the potential on the bulk and the brane tensions are not BPS. This would arise if the supersymmetry was spontaneously broken, generating for instance some cosmological constant on the brane. If the interaction terms on the brane are not the same as the interaction term in the bulk given by the potential $\mathscr{V}(\Phi_I)$, the vacuum expectation value of the scalar fields will be y-dependent. This is in particular the idea proposed by W. Goldberger and M. Wise [116, 122, 123]. Their stabilisation mechanism relies on the assumption that the bulk potential includes quadratic interaction terms whereas some quartic interaction terms are localised on the branes. The classical solutions for these scalar fields will depend on the fifth dimension coordinate y. After integrating over the fifth dimension, one obtains an effective potential U for the radion (the potential is indeed a function of the distance d between the branes). For the special case of the Goldberger and Wise stabilisation mechanism, the resulting effective potential has a minimum for a given value of d. However in the Goldberger and Wise mechanism, the bulk potential and the brane tensions are introduced by hand in the five-dimensional theory. In a more realistic model, details of the higher dimensional physics should be studied in order to deduce the exact form of the bulk potential and the brane tensions as in done in the models of brane inflation presented in the introduction of this thesis. Unless a realistic model for the origin of the bulk potential is proposed, very little can be deduced on the effective four-dimensional potential U which represents interaction between the two branes. Despite this fact, several models have considered the possibility of using this potential U to generate a scale-invariant power spectrum on the branes, corresponding to a scenario of inflation for which the inflaton may be interpreted as the radion. From a four-dimensional point of view, the five-dimensional effects generating interaction between the branes could be interpreted as the presence of a potential in the four-dimensional action (2.56)

$$S = \frac{L}{\kappa} \int \mathrm{d}^4 x \sqrt{-\bar{g}} \left(\frac{1}{2} \bar{R} - \frac{1}{2} \left(\partial \phi \right)^2 - U(\phi) \right), \qquad (2.82)$$

for simplicity we have assumed no gauge and matter fields to be present on the branes. In that case the equation of motion for the scalar field is modified to:

$$\bar{\Box}\phi = U'(\phi), \tag{2.83}$$

and the Einstein equation in Einstein frame becomes:

$$\bar{G}_{\mu\nu} = \partial_{\mu}\phi\partial_{\nu}\phi - \left(\frac{1}{2}(\partial\phi)^2 + U\right)\bar{g}_{\mu\nu}.$$
(2.84)

We therefore have the theory of gravity with a minimally coupled scalar field with potential U. If U satisfies the normal slow-roll constraints for inflation, ϕ might be interpreted as an inflaton scalar field: its perturbations around a quasi-de Sitter background might have generated the observed large scale structure. This gives a natural physical origin for the inflaton which is in general missing in inflationary scenarios [65].

We can now go back to the positive-tension brane frame by performing the conformal transformation (2.51) with the relation (2.53) between ϕ and d: $\left(\cosh\left(\phi/\sqrt{6}\right)\right)^2 = \left(1 - e^{-2d/L}\right)^{-1}$. The theory on the brane is modified to:

$$G_{\mu\nu}^{(+)} = \frac{2}{L} \frac{\Psi^2}{1 - \Psi^2} \Big[D_{\mu} D_{\nu} d - \Box d \, q_{\mu\nu}^{(+)} + \frac{1}{L} \left(\partial_{\mu} d \, \partial_{\nu} d + \frac{1}{2} \left(\partial d \right)^2 q_{\mu\nu}^{(+)} \right) \Big] - U \left(\phi(d) \right) \, \left(1 - \Psi^2 \right) q_{\mu\nu}^{(+)}$$

$$(2.85)$$

$$\Box d = \frac{(\partial d)^{2}}{L} - \frac{L}{\sqrt{6}} \frac{(1 - \Psi^{2})^{2}}{\Psi} U'(\phi)$$

= $\frac{(\partial d)^{2}}{L} + \frac{L^{2}}{6\Psi^{2}} (1 - \Psi^{2})^{3} \frac{\partial}{\partial d} U(\phi(d)),$ (2.86)

where all covariant derivatives and index raising is taken with respect to $q_{\mu\nu}^{(+)}$. The modification of the equation of motion of the scalar field (2.86) ensures the right hand side of the Einstein equation (2.85) to stay conserved even after the addition of a potential.



Figure 2.2: Randall Sundrum model with a bulk brane not submitted any \mathbb{Z}_2 -reflection symmetry.

2.8.2 Bulk Branes

The RS simple structure comes from the fact that the bulk is empty apart from a negative cosmological constant. In the previous subsection we considered a possible extension of this model by studying the effect of bulk scalar fields. Another possible generalisation could come from the presence of bulk branes (as opposed to orbifold branes), Cf. fig. 2.2. Such a bulk brane would not be subjected to any reflection symmetry and could be relevant for the study of D-branes or M-branes in heterotic M-theory. Although realistic D-branes or M-branes have charges and associated form fields that we ignore in the RS picture, they share the same property of the bulk brane to have arbitrary location in the higher-dimension bulk and, as opposed to the boundary branes, not to be constrained by any reflection symmetry.

Background behaviour

We denote by \mathcal{R}_L the region between the positive tension brane and the bulk brane: $Y_+ < Y < Y_b$ and by \mathcal{R}_R the one between the bulk brane and the negative boundary one: $Y_b < Y < Y_-$, where the boundary branes are located at $Y = Y_{\pm}(T)$ and the bulk brane at $Y = Y_b(T)$. In what follows, L, R indices will designate quantities with respect to $\mathcal{R}_{L,R}$. We consider the bulk to have a negative cosmological constant $\Lambda_{L,R} =$ $-6/\kappa L_{L,R}^2$. Assuming cosmological symmetry the five-dimensional geometry can be seen to be SAdS with parameter $\mathcal{C}_{L,R}$ associated with the Black Hole mass on each region. The five-dimensional geometry may be written as

$$ds^{2} = -a^{2}(Y,T) dT^{2} + n^{2}(Y,T) d\mathbf{x}^{2} + dY^{2}, \qquad (2.87)$$

with, in the different regions $\mathcal{R}_{L,R}$,

$$a^{2}(Y,T) = \alpha_{L,R} \left(e^{-2(Y-Y_{b}(T))/L_{L,R}} + \frac{\mathcal{C}_{L,R}}{4} e^{2(Y-Y_{b}(T))/L_{L,R}} \right)$$
(2.88)

$$n^{2}(Y,T) = \beta_{L,R} \left(a^{2} - \frac{\mathcal{C}_{L,R} \alpha_{L,R}^{2}}{a^{2}} \right),$$
 (2.89)

and for simplicity we assumed the branes to be spatially flat. In order to keep the fivedimensional metric continuous across the bulk brane, we impose $\alpha_{L,R} = a_b^2 (1 + C_{L,R}/4)^{-1}$, and $\beta_{L,R} = n_b^2 (a_b^2 - C_{L,R}\alpha_{L,R}^2/a_b^2)^{-1}$, where a_b is the value of the scale factor on the bulk brane and similarly for n_b .

We now suppose that the bulk brane tension is fine-tuned to its canonical value

$$\lambda_b = \frac{3}{\kappa} \left(\frac{1}{L_R} - \frac{1}{L_L} \right). \tag{2.90}$$

From the Israël matching condition (2.10) on the bulk brane, we have:

$$\Delta K_{j}^{i}(Y_{b}) = K_{j}^{i}(Y_{b}^{+}) - K_{j}^{i}(Y_{b}^{-}) = -\frac{\kappa}{3} (\lambda_{b} + \rho_{b}) \delta_{j}^{i}, \qquad (2.91)$$

with ρ_b the energy density of matter fields located on the brane and

$$K_{j}^{i}(Y_{b}^{\pm}) = -\delta_{j}^{i} \sqrt{\frac{1}{L_{R,L}^{2}} - \frac{\mathcal{C}_{R,L}}{L_{R,L}^{2} a_{b}^{4}} + H_{b}^{2}}.$$
(2.92)

with $H_b = \frac{1}{a_b^2} \frac{\mathrm{d}a(Y_b)}{\mathrm{d}\tau}$. It is worth pointing out that from the five-dimensional point of view, a_b is a constant, but if the bulk brane is moving, $Y_b = Y_b(T)$, the scale factor on the brane

is not constant. Solving (2.91) with the background expression (2.92) for the extrinsic curvature, we obtain the modified Friedmann equation on the bulk brane:

$$H_b^2 = \frac{1}{4} \left(\frac{\kappa}{3} \left(\rho_b + \lambda_b \right) \right)^2 + \frac{1}{4 \left(\frac{\kappa}{3} \left(\rho_b + \lambda_b \right) \right)^2} \left[\frac{1}{L_L^2} - \frac{1}{L_R^2} - \frac{1}{a_b^4} \left(\frac{\mathcal{C}_L}{L_L^2} - \frac{\mathcal{C}_R}{L_R^2} \right) \right]^2 (2.93)$$
$$- \frac{1}{2} \left[\frac{1}{L_L^2} + \frac{1}{L_R^2} - \frac{1}{a_b^4} \left(\frac{\mathcal{C}_L}{L_L^2} + \frac{\mathcal{C}_R}{L_R^2} \right) \right].$$

As mentioned in previous works [124], the absence of \mathbb{Z}_2 -symmetry gives rise to additional term in the induced Friedmann equation of the bulk brane. Besides the presence of the dark energy term and the quadratic term in the energy density which have already been noticed for boundary branes, the Friedmann equation of a generic bulk brane contains as well a term of the form a^{-8} which can not be interpreted as any kind of physical matter. Furthermore matter couples to gravity in a very peculiar way as it can be seen from the presence of the $\left(\frac{\kappa}{3} \rho_b + \lambda_b\right)^{-2}$ term.

If a reflection symmetry had been imposed around the bulk brane, the relation $L_R = -L_L$ and $C_L = C_R$ should have hold and the extra asymmetric term would have cancelled: $H_b^2 = \frac{\kappa}{6} \lambda_b \rho_b + \frac{\kappa^2}{36} \rho_b^2 + \frac{C_R}{L_R^2 a_b^4}$ which is precisely what we obtain for a \mathbb{Z}_2 -symmetric orbifold brane. Furthermore, we can notice that this extra term are of higher order in velocity, such term will therefore be absent at low energies. However they represent characteristic terms of braneworld cosmology which we believe are worth studying.

Covariant formalism

In what follows, we will consider the Gauss-Codacci formalism in order to understand the origin of the extra asymmetric terms present in (2.93). For simplicity we shall first consider an empty brane. Staring from the Israël Matching condition (2.10), the key difference in the extrinsic curvature of a brane with or without \mathbb{Z}_2 -symmetry can be emphasised. We can then give a brief overview of what happens when matter is confined on the brane. The extrinsic curvature on each side of the branes can be expressed as

$$K^{\mu}_{\nu}(Y^{\pm}_{b}) = -\frac{1}{L_{R,L}} \delta^{\mu}_{\nu} + \bar{K}_{\mu\nu}. \qquad (2.94)$$

When the brane is submitted to a reflection symmetry, the "average"-tensor (or "asymmetric"tensor) $\bar{K}^{\mu}_{\nu} = \frac{1}{L_R - L_L} \left(L_R K^{\mu}_{\nu}(Y^+_b) - L_L K^{\mu}_{\nu}(Y^-_b) \right)$ vanishes but for a bulk brane this is not necessarily the case. Using the Gauss equation (2.12), the Ricci tensor may be expressed on each side of the brane by:

$$R^{\mu}_{\nu}(Y^{\pm}_{b}) = -\frac{2}{L_{R,L}} \left(\bar{K}^{\mu}_{\nu} + \frac{1}{2} \bar{K} \,\delta^{\mu}_{\nu} \right) + \left(\bar{K} \,\bar{K}^{\mu}_{\nu} - \bar{K}^{\mu}_{\alpha} \bar{K}^{\alpha}_{\nu} \right) - E^{\mu}_{\nu}(Y^{\pm}_{b}). \tag{2.95}$$

The Weyl tensor $E_{\mu\nu}$ being traceless,

$$R(Y_b^{\pm}) = -\frac{6}{L_{R,L}} \bar{K} + \left(\bar{K}^2 - \bar{K}_{\alpha\beta}\bar{K}^{\alpha\beta}\right).$$
(2.96)

The metric has to be continuous across the brane junction and so does the Ricci tensor. Imposing $R(Y_b^+) = R(Y_b^-)$, the average part of the extrinsic curvature must be traceless $\bar{K} = 0$. Furthermore, from the Codacci equation (2.8), $\bar{K}_{\mu\nu}$ should be transverse. The "average"- tensor $\bar{K}_{\mu\nu}$ for an empty brane is therefore transverse and traceless. The Ricci tensor will be well-defined on the brane only if the Weyl tensor satisfies:

$$E^{\mu}_{\nu}(Y^{\pm}_{b}) = \frac{2}{L_{R,L}} \bar{K}^{\mu}_{\nu} + \bar{E}^{\mu}_{\nu}, \qquad (2.97)$$

with the tracelessness of $E_{\mu\nu}$ implying $\bar{E}^{\mu}_{\mu} = 0$. For an empty brane, the Gauss equation therefore simplifies to:

$$R^{\mu}_{\nu} = \bar{K}^{\mu}_{\alpha} \bar{K}^{\alpha}_{\nu} - \bar{E}^{\mu}_{\nu}, \qquad (2.98)$$

where $\bar{K}_{\mu\nu}$ is transverse and traceless. For the background, it therefore looks like the stress-energy tensor of radiation, exactly as the Weyl tensor $E_{\mu\nu}$ (2.29). However $\bar{K}_{\mu\nu}$ comes in quadratically in the modified Einstein equation and therefore leads to the extra a_b^{-8} term in the modified Friedman equation (2.93). This is a property of any brane not submitted to any reflection symmetry. In a low-energy theory, these terms are expected to be negligible, but the purpose of this thesis is to go beyond the low-energy limit and therefore to understand the way these terms might affect cosmology in such a brane. In the presence of matter, the Gauss equation on the brane becomes much more complicated as it presents some quadratic terms in both $\bar{K}_{\mu\nu}$ and in the stress-energy tensor for matter fields on the brane. The resulting "average"-tensor $\bar{K}_{\mu\nu}$ is therefore not traceless and transverse anymore but satisfies:

$$T^{\alpha\beta}\bar{K}_{\alpha\beta} - \lambda_b \bar{K} = \frac{1}{2} \left(\frac{1}{L_R} + \frac{1}{L_L} \right) T$$
(2.99)

$$D_{\mu}\bar{K}^{\mu}_{\nu} = \bar{K}_{,\nu}.$$
 (2.100)

Here we notice a remarkable fact, when non-conformal matter is present on the bulk brane, the average tensor $\bar{K}_{\mu\nu}$ can not cancel if $L_R \neq L_L$, *ie.* if there is no reflection symmetry between \mathcal{R}_L and \mathcal{R}_R , having non-conformal matter on the brane imposes the presence of an asymmetric term in the brane. The Modified Einstein equation on the brane is then

$$R^{\mu}_{\nu} = \frac{\kappa^2}{4} \tilde{T}^{\mu}_{\alpha} \tilde{T}^{\alpha}_{\nu} + \bar{K}^{\mu}_{\alpha} \bar{K}^{\alpha}_{\nu} - \bar{E}^{\mu}_{\nu}, \qquad (2.101)$$

where $\tilde{T}_{\mu\nu} = T_{\mu\nu} - \frac{1}{3}Tq_{\mu\nu}$. This result seems at first sight very surprising as there does not appear to be any linear term in $T_{\mu\nu}$ as would be expected in a usual theory of gravity. However the linear terms in $T_{\mu\nu}$ come in through the quadratic terms in $\bar{K}_{\mu\nu}$ which depend on $T_{\mu\nu}$ through (2.99) with the condition (2.100). It is a priori very difficult to make much analytical progress on these equations when dealing with covariant quantities and not in a specific scenario. For the background, we have shown that the modified Friedmann equation was genuinely very different from standard gravity and we have here presented the covariant approach to point out the particularities of such a model.

2.9 Discussion

The Randall Sundrum model is a remarkable toy model on which several features of braneworld cosmology can be tested. Despite its simplicity, a number of departures from standard four-dimensional gravity can be observed. First the coupling of gravity to matter is genuinely different, and becomes very peculiar for a bulk brane not submitted to any reflection symmetry. Another difference arises from the presence of Kaluza Klein modes on the branes which represents bulk excitations and are generated on the branes through the induced Weyl tensor. At low energies, an effective theory can be derived on the branes. In that case gravity behaves in an essentially four-dimensional way and couples to a scalar field which can be interpreted as the distance between the branes. However most of the characteristic features of the RS model are left out by this low-energy effective theory such as the Kaluza Klein corrections, and the peculiar coupling to matter. Despite this fact, numerous work has been done on the derivation and the use of this effective theory. We have here chosen to show its derivation in a different way from what is usually found in the literature, in order to understand better where specific approximations have been made and how extension can be studied, which is the main motivation of this thesis. In the next chapter we shall therefore start by showing how the low-energy theory can be extended in order to model correctly the way gravity couples to matter and see how this could potentially affect cosmology.

CHAPTER 2. BRANEWORLD COSMOLOGY

Chapter 3

Quadratic Terms in the Stress-energy



Itzam Ná, God of the Sky (Maya)

3.1 Introduction

In this chapter, we shall concentrate on braneworld models for which the energy scales are important in comparison to the brane tension. As we have seen in the previous chapter, their gravitational behaviour is then expected to be genuinely different from conventional four-dimensional gravity. From the cosmological point of view, this aspect is of great interest as it may potentially provide new scenarios for the beginning of our Universe. Although we expect the conventional low-energy limit of braneworld gravity to be valid at the present time, the low-energy approximation might well be violated in the early Universe. This will, for instance, be the case if the energy scale of inflation is comparable to the brane tension scale. In such a scenario, the Friedmann equation just after the Big Bang, might thus be modified. Additional terms such as the dark radiation or the ρ^2 terms (where ρ is the energy density of matter on the brane) can then play an important rôle.

Unfortunately, working beyond the low-energy limit usually requires a five-dimensional description [125,126], making analytic solutions very hard to follow or requiring numerical methods, (unless some very restrictive conditions are imposed). For the study of perturbations, and in particular their possible interpretation as generating the observed large scale structure, the analysis would be facilitated if it could be reexpressed in a four-dimensional language [127, 128]. In order to do so, and to get a better insight into the high-energy regime, it is useful to use approximation methods that capture the essential physics, if not the precise solution. In this chapter, we will focus on the adiabatic approximation, allowing us to go beyond the low-energy restriction. Using this approximation, we shall see that the same four-dimensional tools can still be used for the study of perturbations.

As already mentioned in chapter 2, we may use the Gauss-Codacci formalism to derive the modified Einstein equation on a brane. However, from the brane four-dimensional point of view, the five-dimensional nature of the model cannot be completely ignored. Indeed, the system of equations obtained on the brane is not closed since the modified Einstein equation contains a new term: the electric part $E_{\mu\nu}$ of the five-dimensional Weyl tensor. This term encodes information about the bulk geometry, which cannot be solved exactly unless one assumes cosmological symmetry or considers other specific cases. To start with, the exact solution for the background may be used to derive a covariant expression for $E_{\mu\nu}$. Formally, this expression is only valid for the background, but for linear perturbations in an adiabatic regime, we may argue that the main part of this expression remains accurate. Furthermore, when this formalism is applied to the two-brane RS model, an important assumption can be made: This work will rely on the assumption that the metrics on both branes are conformal to each other. The underlying idea behind this formalism can actually be applied to a wide range of braneworld scenarios. The same concept may for instance be used to study the asymmetric term present on a generic bulk brane. As another example, we may consider the one-brane limit of the RS model, bringing the negative tension brane to infinity and leaving the positive tension brane embedded in an infinite extra-dimension. In this limit, no conformal assumption is hence required.

In this special one-brane limit, this formalism is used within the context of brane inflation as studied in [129–132]. The inflaton scalar field is confined on the brane and evolves in a potential which should satisfy some slow-roll constraints [131,133,134]. In the one-brane limit, the negative tension brane does not backreact on the positive one and the radion does not couple with the inflation scalar field introduced on the brane, the physics will hence be straight-forward. However, in the context of braneworld inflation, the constraints on the potential are expected to be different from the potential of chaotic inflation and can indeed be computed. In order to do so, the scalar and tensor perturbations are evaluated and we give an estimate of the non-gaussianities. When the slow-roll conditions are imposed, the model reproduces the nearly scale-invariant spectrum for density perturbations of standard inflation. It is hence possible to show how the main features of brane inflation may generally be reinterpreted as arising from an ordinary inflation model with a redefined potential, or equivalently with redefined slow-roll parameters.

3.2 Covariant Treatment

3.2.1 Two-brane Formalism

Conformal Assumption

The aim of this section is to formulate a covariant treatment for the quadratic terms in the stress-energy. These terms are indeed present in the induced Einstein equation of a brane (2.19) and we imagine a scenario for which their contribution could be important. Our starting point is the assumption that, for long wavelength adiabatic perturbations, the metrics on both branes remain conformally related

$$q_{\mu\nu}^{(-)} = \Psi^2 q_{\mu\nu}^{(+)}.$$
(3.1)

This assumption is clearly trivial for cosmological symmetry: Assuming such a symmetry, the background metric on the branes have indeed no other choice than being conformal to each other. This relation is as well valid in the low-energy effective theory (2.37), however the relation between the conformal factor Ψ and the proper distance d between the branes does not need to be the same as the one fixed in that limit (2.38). In a general case, we will assume Ψ to be a generic function of d, and of any of its derivatives. For a SAdS background for instance, as in (2.20), the conformal factor between the branes will be a function of the Black Hole mass associated with C, which in turn can be expressed in terms of derivatives of the distance between the branes.

However, one might worry that this conformal relation should not hold in general. We have indeed seen in section 2.7, that beyond the cosmological symmetric case, the metric on both branes does not remain conformal to each other at the perturbed level. In the quasi-static case, for instance, a five-dimensional treatment shows that the induced metric on both branes (2.73) and (2.74) is not conformal beyond the zero mode, and this is true independently of the proper distance between them. However, we may stress that for this quasi-static case, the zero mode was well described by the low-energy effective theory of section 2.6 since the branes were static and empty for the background. If on the other hand, the stress-energy of matter fields living on the brane is important, the low-energy effective theory will not be an acceptable model, not even to describe the zero mode. This is the reason why in this chapter we focus on the study of the zero mode beyond the lowenergy approximation. This treatment will not reproduce all the general features present in braneworld cosmology, but we will see that it represents a good starting point for the study of the quadratic terms in the stress-energy. In the next chapter, this treatment can then be extended, and in particular, we shall then see how this formalism may be modified in order to introduce corrections coming from the KK modes (which are indeed missing here). In that case, the conformal assumption (3.1) should hence be modified.

Properties of the Weyl tensor

Using this conformal assumption (3.1), the Ricci tensor on the negative tension brane can be expressed in terms of the Ricci tensor on the positive one:

$$\Psi^2 R^{(-)}_{\mu\nu} = \Psi^2 R^{(+)}_{\mu\nu} - 2\Psi D_{\mu} D_{\nu} \Psi + 4 \,\partial_{\mu} \Psi \,\partial_{\nu} \Psi - q^{(+)}_{\mu\nu} \left(\Psi \Box \Psi + (\partial \Psi)^2 \right), \tag{3.2}$$

where all derivatives and contractions are taken with respect to the metric $q_{\mu\nu}^{(+)}$ on the positive tension brane. This will be our convention throughout this section unless otherwise specified. The trace of equation (3.2) then provides the equation of motion of the scalar field Ψ :

$$\Box \Psi = \frac{1}{6} \left(R^{(+)} - \Psi^2 R^{(-)} \right) \Psi, \qquad (3.3)$$

where $R^{(-)} = q^{(-)\mu\nu}R^{(-)}_{\mu\nu}$. We may notice that if the negative tension brane is empty, $R^{(-)} = 0$ (as a consequence of (2.19) and of the tracelessness of the Weyl tensor). In that case, the scalar field Ψ will hence appear to be conformally invariant with respect to the positive tension brane metric: $\Box \Psi = \frac{1}{6}R^{(+)}\Psi$, and conversely if the positive tension brane is empty Ψ^{-1} will be conformally invariant with respect to the negative one.

As already mentioned in section 3.2.2, the only unknown quantity in the projected Einstein equations on each brane is the Weyl tensor $E_{\mu\nu}^{(\pm)}$ as defined in (2.4). Although $E_{\mu\nu}^{(\pm)}$ is not determined from the brane point of view, its exact expression can be derived for the background and may be expressed as in (2.29). For the background, $E_{\mu\nu}^{(-)}$ and $E_{\mu\nu}^{(+)}$ are therefore conformally related:

$$\Psi^2 E^{(-)}_{\mu\nu} = E^{(+)}_{\mu\nu}, \qquad (3.4)$$

as predicted by the low-energy effective theory in (2.42), although we recall that the relation between Ψ and d is not necessarily the one predicted by (2.38). Covariantly, however, the only property which is known for the Weyl tensor is the requirement that $E^{(\pm)}_{\mu\nu}$ is traceless and that its divergence must be consistent with the Bianchi identity on each brane $D^{(\pm)}_{\mu}G^{(\pm)}_{\ \nu}=0.$ Using the expression (2.19) for the induced Ricci tensor on each brane, we can construct the quantity $\left(R_{\mu\nu}^{(+)} - \Psi^2 R_{\mu\nu}^{(-)}\right)$:

$$\left(R_{\mu\nu}^{(+)} - \Psi^2 R_{\mu\nu}^{(-)}\right) = \left(\Pi_{\mu\nu}^{(+)} - \Psi^2 \Pi_{\mu\nu}^{(-)}\right) - \Delta E_{\mu\nu},\tag{3.5}$$

where we have used the notation

$$\Pi_{\mu\nu}^{(\pm)} = \pm \frac{\kappa}{L} \left(T_{\mu\nu}^{(\pm)} - \frac{1}{2} T^{(\pm)} q_{\mu\nu}^{(\pm)} \right) - \frac{\kappa^2}{4} \left(T_{\mu\nu}^{(\pm)\alpha} T_{\alpha\nu}^{(\pm)} - \frac{1}{3} T^{(\pm)} T_{\mu\nu}^{(\pm)} \right).$$
(3.6)

Much of the following work will rely on the fact that $\Delta E_{\mu\nu}$ vanishes for the background, which is simply a consequence of (3.4). Covariantly, $\Delta E_{\mu\nu}$ is indeed defined as:

$$\Delta E_{\mu\nu} = \left(E_{\mu\nu}^{(+)} - \Psi^2 E_{\mu\nu}^{(-)} \right).$$
(3.7)

Since the brane metrics are conformal to each other, the traceless property of $E_{\mu\nu}^{(\pm)}$ can be extended to $\Delta E_{\mu\nu}$:

$$q^{(+)\,\mu\nu}\,\Delta E_{\mu\nu} = 0. \tag{3.8}$$

Using the covariant relation between $R_{\mu\nu}^{(-)}$ and $R_{\mu\nu}^{(+)}$ (3.2), the Ricci tensor on the positive tension brane may now be expressed in terms of the scalar field Ψ :

$$R_{\mu\nu}^{(+)} = \Pi_{\mu\nu}^{(+)} + U_{\mu\nu}^{\Psi} - \Delta E_{\mu\nu}, \qquad (3.9)$$

where the tensor $U^{\Psi}_{\mu\nu}$ satisfies:

$$U^{\Psi}_{\mu\nu} = -2\Psi D_{\mu}D_{\nu}\Psi + 4\,\partial_{\mu}\Psi\,\partial_{\nu}\Psi - q^{(+)}_{\mu\nu}\left(\Psi\Box\Psi + (\partial\Psi)^{2}\right) + \Psi^{2}R^{(+)}_{\mu\nu} - \Psi^{2}\Pi^{(-)}_{\mu\nu} \tag{3.10}$$

$$q^{(+)\mu\nu}U^{\Psi}_{\mu\nu} = 0 \tag{3.11}$$

$$D^{(+)}_{\mu}U^{\Psi}{}^{\mu}_{\nu} = -\Psi^2 D^{(-)}_{\mu} \breve{\Pi}^{(-)}{}^{\mu}_{\nu}.$$
(3.12)

For any tensor Z^{μ}_{ν} , we define the associated tensor \breve{Z}^{μ}_{ν} such as $\breve{Z}^{\mu}_{\nu} = Z^{\mu}_{\nu} - \frac{1}{2}Z^{\alpha}_{\alpha}\delta^{\mu}_{\nu}$. For the Bianchi identity to be satisfy on the positive brane, the divergence of $\Delta E_{\mu\nu}$ is constrained to be:

$$D_{\mu}^{(+)}\left(q^{(+)\mu\alpha}\Delta E_{\alpha\nu}\right) = D_{\mu}^{(+)}\breve{\Pi}^{(+)\,\mu}_{\ \nu} - \Psi^2 D_{\mu}^{(-)}\breve{\Pi}^{(-)\,\mu}_{\ \nu}.$$
(3.13)

Decomposition of the Weyl tensor

In the following, we shall decompose the traceless tensor $\Delta E_{\mu\nu}$ into a vector part and a tensor (transverse) part. In other world, we shall assume, the inhomogeneous part of $\Delta E_{\mu\nu}$ to be absorbed into its vector part:

$$\Delta E_{\mu\nu} = \mathcal{E}_{\mu\nu} + E_{\mu\nu}^{TT}, \qquad (3.14)$$

with the vector part

$$\mathcal{E}_{\mu\nu} = D^{(+)}_{\mu} \mathcal{E}_{\nu} + D^{(+)}_{\nu} \mathcal{E}_{\mu} - \frac{1}{2} q^{(+)}_{\mu\nu} D^{(+)}_{\alpha} \mathcal{E}^{\alpha}, \qquad (3.15)$$

and the homogeneous part $E^{TT\mu}_{\nu}$ satisfying the relations:

$$D^{(+)}_{\mu}E^{TT\,\mu}_{\ \nu} = 0 \quad \text{and} \quad E^{TT\,\mu}_{\ \mu} = 0.$$
 (3.16)

The vector $\mathcal{E}_{\mu} = \mathcal{E}_{\mu}^{(+)} - \mathcal{E}_{\mu}^{(-)}$ can then be determined using the divergence relation (3.13) for ΔE_{ν}^{μ} :

$$\mathcal{E}_{\mu\nu} = \mathcal{E}_{\mu\nu}^{(+)} - \mathcal{E}_{\mu\nu}^{(-)}, \qquad (3.17)$$

$$D^{(+)}_{\mu} \mathcal{E}^{(+)\,\mu}_{\ \nu} = D^{(+)}_{\mu} \breve{\Pi}^{(+)\,\mu}_{\ \nu}, \qquad (3.18)$$

$$D^{(+)}_{\mu} \mathcal{E}^{(-)\,\mu}_{\ \nu} = \Psi^2 D^{(-)}_{\mu} \breve{\Pi}^{(-)\,\mu}_{\ \nu}.$$
(3.19)

Although this determines the vector part only up to some integration constants (or more precisely up to a homogeneous part), we may for instance avoid this problem by assuming that all the divergenceless parts of $\Delta E_{\mu\nu}$ are contained in $E_{\mu\nu}^{TT}$. More precisely, the vector part will be uniquely determined if the condition $D_{\mu}\mathcal{E}^{(\pm)}{}^{\mu}{}_{\nu} = 0 \Leftrightarrow \mathcal{E}^{(\pm)}{}_{\mu}{}_{\nu} = 0$ was imposed as well with some initial conditions. Adding the contribution of $\mathcal{E}^{(-)}{}^{\mu}{}_{\nu}$ to $U_{\mu\nu}^{\Psi}$, we obtain the conserved and traceless pseudo-stress-energy tensor $T_{\mu\nu}^{\Psi \text{ eff}}$:

$$T^{\Psi \text{ eff}}_{\mu\nu} = U^{\Psi}_{\mu\nu} + \mathcal{E}^{(-)}_{\mu\nu}, \qquad (3.20)$$

$$T^{\Psi \text{ eff}}_{\mu\nu} = 4 \,\partial_{\mu}\Psi \,\partial_{\nu}\Psi - 2\Psi D_{\mu}D_{\nu}\Psi + q^{(+)}_{\mu\nu} \left(2\Psi \Box \Psi - (\partial\Psi)^{2}\right)$$

$$+ G^{(+)}\Psi^{2} - \Psi^{2}\breve{\Pi}^{(-)} + \mathcal{E}^{(-)}$$
(3.21)

$$q^{(+)\mu\nu}T^{\Psi \text{ eff}}_{\mu\nu} = 0, \qquad (3.22)$$

$$D^{(+)}_{\mu}T^{\Psi} {}^{\text{eff}\,\mu}_{\nu} = 0.$$
(3.23)

When the negative tension brane is empty, we may note that $\Psi^2 \breve{\Pi}_{\mu\nu}^{(-)} = \mathcal{E}_{\mu\nu}^{(-)} = 0$. Using the relation (3.3), $T_{\mu\nu}^{\Psi \text{eff}}$ is then precisely the stress-energy tensor for the conformally invariant scalar field Ψ . The tensor $T_{\mu\nu}^{\Psi \text{eff}}$ can hence be seen as the natural extension of the stress-energy tensor for the scalar field to the case where matter is present on the negative tension brane. In particular, the traceless and divergenceless properties of the pseudo-stress-energy tensor are still satisfied. For this to remain valid, some contributions had been added to the stress-energy tensor and accommodate the fact that Ψ is no longer a conformally invariant scalar field with respect to the positive tension brane.

In the modified Einstein equation (3.9), the only part that remains undetermined is the traceless and divergenceless contribution $E_{\mu\nu}^{TT}$ to $\Delta E_{\mu\nu}$. However, from the relation (3.4), we may recall that $\Delta E_{\mu\nu} = E_{\mu\nu}^{(+)} - \Psi^2 E_{\mu\nu}^{(-)} = 0$ for cosmological symmetry. Furthermore, we may check that for any kind of matter satisfying conservation of energy in a homogeneous and isotropic background, the divergence of $\breve{\Pi}^{(\pm)\mu}_{\ \nu}$ in (3.18, 3.19) vanishes. Hence both vector parts $\mathcal{E}_{\mu\nu}^{(\pm)}$ vanish for the background, and we may point out that this result remains valid even when the bulk geometry is not taken to be pure AdS ($\mathcal{C} \neq 0$). The remaining part $E_{\mu\nu}^{TT}$ is hence bounded to vanish for the background as well. For the purpose of long wavelength adiabatic perturbations, it is therefore consistent to neglect any contribution from $E_{\mu\nu}^{TT}$, as it will be argued in what follows.

Adiabaticity argument

It is a well-known result that for long wavelength *adiabatic* scalar perturbations, the quantity $\delta \rho / \dot{\rho}$ (where ρ is the energy density) is the same for any conserved fluid, regardless of its equation of state [10]. In particular, this is true for the conserved tensor $E_{\mu\nu}^{TT}$ (with density ρ_E and $\omega_E = 1/3$). The transverse condition implies $\dot{\rho}_E = -4 \frac{\dot{a}}{a} \rho_E$. Comparing ρ_E with the energy density ρ of any other fluid present in the theory, the adiabaticity condition at long wavelengths imposes $\delta \rho_E / \dot{\rho_E} \approx \delta \rho / \dot{\rho}$ which may be expressed as:

$$\delta \rho_E \approx -4 \,\rho_E \frac{\dot{a}}{a} \frac{\delta \rho}{\dot{\rho}}.\tag{3.24}$$

Since the tensor $E_{\mu\nu}^{TT}$ vanishes for the background, $\rho_E = 0$, the perturbations are hence bounded to do the same $\delta\rho_E \approx 0$. In an adiabatic regime, the long wavelength scalar perturbations of $E_{\mu\nu}^{TT}$ can thus be neglected.

For the three-dimensional-tensor perturbations, the same approximation will be made, although no analogous argument may be given in that case. It is however in the context of this approximation that we shall consider inflation on the brane in the next section.

Covariant formalism for the two-brane model

Thanks to the Gauss-Codacci formalism and to the conformal assumption (3.1), we have therefore shown that it is possible to find an expression for the Weyl tensor $E_{\mu\nu}$ up to a conserved traceless tensor that we shall neglect as a first approximation. Comparing equation (2.19) with (3.9), the expression for the Weyl tensor is

$$E_{\mu\nu}^{(+)} = -T_{\mu\nu}^{\Psi \text{ eff}} + \mathcal{E}_{\mu\nu}^{(+)}.$$
(3.25)

In this regime, the metric on both branes is governed by the following closed set of equations, including the modified Einstein equation on the brane, and the associated Klein Gordon equation for the scalar field:

$$R_{\mu\nu}^{(+)} = \left(\Pi_{\mu\nu}^{(+)} - \mathcal{E}_{\mu\nu}^{(+)}\right) + T_{\mu\nu}^{\Psi \text{ eff}},$$

$$\Box \Psi = \frac{1}{6} \left(R^{(+)} - \Psi^2 \Pi^{(-)}\right) \Psi,$$
(3.26)

with the following definitions:

$$\begin{aligned}
\begin{pmatrix}
q_{\mu\nu}^{(-)} = \Psi^2 q_{\mu\nu}^{(+)} \\
\Pi_{\mu\nu}^{(\pm)} = \pm \frac{\kappa}{L} \left(T_{\mu\nu}^{(\pm)} - \frac{1}{2} T^{(\pm)} g_{\mu\nu}^{(\pm)} \right) - \frac{\kappa^2}{4} \left(T_{\mu}^{(\pm)\alpha} T_{\alpha\nu}^{(\pm)} - \frac{1}{3} T^{(\pm)} T_{\mu\nu}^{(\pm)} \right) \\
T_{\mu\nu}^{\Psi \text{ eff}} = 4\partial_{\mu} \Psi \, \partial_{\nu} \Psi - 2\Psi D_{\mu} D_{\nu} \Psi + g_{\mu\nu}^{(+)} \left(2\Psi \Box \Psi - (\partial\Psi)^2 \right) + \Psi^2 G_{\mu\nu}^{(+)} - \Psi^2 \breve{\Pi}_{\mu\nu}^{(-)} + \mathcal{E}_{\mu\nu}^{(-)} \\
\mathcal{E}_{\mu\nu}^{(\pm)} = D_{\mu}^{(+)} \mathcal{E}_{\nu}^{(\pm)} + D_{\nu}^{(+)} \mathcal{E}_{\mu}^{(\pm)} - \frac{1}{2} q_{\mu\nu}^{(+)} D_{\alpha}^{(+)} \mathcal{E}^{(\pm)\alpha} \\
D_{\mu}^{(+)} \mathcal{E}_{\nu}^{(-)\mu} = D_{\mu}^{(+)} \breve{\Pi}_{\nu}^{(-)\mu} \\
D_{\mu}^{(+)} \mathcal{E}_{\nu}^{(-)\mu} = \Psi^2 D_{\mu}^{(-)} \breve{\Pi}_{\nu}^{(-)\mu}
\end{aligned}$$
(3.27)

where we wrote $\Pi^{(\pm)} = q^{(\pm)\mu\nu}\Pi^{(\pm)}_{\mu\nu}$, and any other contractions and derivatives are performed with respect to $q^{(+)}_{\mu\nu}$.

These may look formidable, we have an equation of motion for a non-minimally coupled scalar field Ψ and equations for the metric $q_{\mu\nu}^{(+)}$ sourced in a highly non-trivial way by both Ψ and the stress-energy on each brane. However, we argue that the system (3.26) with (3.27) is now closed, is self-consistent, and allows us to calculate the long-wavelength adiabatic perturbations when the ρ^2 terms play a significant rôle and cannot be neglected. This resulting model is perfectly consistent with the effective theory derived in the low-energy limit. Indeed, in the limit where $\rho_{\pm} \ll \lambda$, we recover the same theory as in section 2.6.

In the limit where Ψ vanishes, we immediately recover the one-brane RS model, where the brane is embedded in a non compactified extra-dimension [51]. We will study this limit in more detail in the next section, but first we may point out how this technique may be used in the case of a non- \mathbb{Z}_2 brane to model the asymmetric which term generically present, Cf. section 2.8.2.

Characteristic terms of a non- \mathbb{Z}_2 bulk brane

In this subsection, we shall only point out how the previous method can be applied for the study of bulk branes. In particular, we shall apply the same formalism to the asymmetric term $\bar{K}_{\mu\nu}$ which is typically present on bulk branes Cf. eq.(2.94). We have seen in section 2.8.2, that the asymmetric term $\bar{K}_{\mu\nu}$ has some interesting properties when non-conformally invariant matter is present on the bulk brane. However in what follows, we shall focus on the situation for which no matter and gauge fields are confined to the bulk brane. In that case, we have seen from (2.96) that $\bar{K}_{\mu\nu}$ is traceless and, from the Codacci equation, that $\bar{K}_{\mu\nu}$ is transverse. Furthermore, assuming cosmological symmetry, the five-dimensional geometry can be solved and so the exact expression of $\bar{K}_{\mu\nu}$ for the background can be found. Covariantly, $\bar{K}_{\mu\nu}$ may therefore be expressed as a term $\bar{K}^{(0)}_{\mu\nu}$ which gives the correct result for the background plus an extra piece $\bar{K}^{TT}_{\mu\nu}$ which cancels for the background. Bearing in mind that $\bar{K}^{TT}_{\mu\nu}$ is traceless and transverse, it is therefore possible to neglect

its contribution, for adiabatic perturbations. Since this goes beyond the purpose of this chapter, we will leave the details of this computation to the appendix A. In this appendix, we will see that $\bar{K}^{(0)}_{\mu\nu}$ depends both on the distance between the bulk brane and the orbifold ones and on the matter content of these ones, Cf. eq.(A.16). The aim of this subsection was only to illustrate the fact that this method could be used in a number of different scenarios in braneworld cosmology. In what follows, we shall concentrate our study to the one-brane RS limit.

3.2.2 One-brane Limit

We shall now consider a special case that will allow us to focus on the effect of the quadratic terms in $T_{\mu\nu}$. In particular, we choose the background geometry to be purely AdS (C = 0in (2.20)). In that case, each brane evolves independently from each other in the background since the Weyl tensor vanishes for the background, and hence no information can be transmitted through the bulk. The distance between the brane, has no impact on the brane, and in particular, the negative tension brane could be very close or could be sent to infinity without noticing any differences on the positive brane. When the distance between the brane is infinite, we actually recover the one-brane RS model for the positive tension brane, which is the limit we shall focus on in what follows. Since we will be interested in quantities on the positive tension brane only, the superscript (+) will be suppressed.

Following the same decomposition for the Weyl tensor as in the previous subsection, in (3.14), we may write the Weyl tensor on the positive tension brane as:

$$E_{\mu\nu} = \mathcal{E}_{\mu\nu}^{(L)} + E_{\mu\nu}^{TT}, \qquad (3.28)$$

$$D_{\mu}E^{TT\,\mu}_{\ \nu} = 0, \quad E^{TT\,\mu}_{\ \mu} = 0. \tag{3.29}$$

Furthermore, since the Bianchi identity has to be satisfied for the four-dimensional brane Einstein tensor, the divergence of the Weyl tensor must be:

$$D_{\mu}E^{\mu}_{\nu} = -\frac{\kappa^2}{4}T^{\beta}_{\alpha}\left(D_{\beta}(T^{\alpha}_{\nu} - \frac{1}{3}T\delta^{\alpha}_{\nu}) - D_{\nu}(T^{\alpha}_{\beta} - \frac{1}{3}T\delta^{\alpha}_{\beta})\right),$$
(3.30)

$$=D_{\mu}\mathcal{E}^{(L)\,\mu}_{\quad\nu}.\tag{3.31}$$

The longitudinal part may be determined up to a homogeneous tensor which is absorbed in the transverse and traceless part $E_{\mu\nu}^{TT}$. As already mentioned, the divergence in (3.30) vanishes for any kind of matter satisfying conservation of energy in a homogeneous and isotropic background, the part contributing to the Weyl tensor in (2.29) is hence homogeneous and $D_{\mu}E_{\nu}^{\mu} = 0$ for the background.

As before, $E_{\mu\nu}^{TT}$ is the only quantity that remains undetermined, but for the purpose of this chapter, we will make the important assumption that $E_{\mu\nu}^{TT}$ can be neglected. For a pure AdS background, the Weyl tensor vanishes and $E_{\mu\nu}^{TT}$ only comes in at first order in perturbations. Corrections arising from this term will be considered in the next chapter, but for now, we only focus on the behaviour of the quadratic terms in the stress-energy. We can hence use this approach to study inflation on the positive tension brane and understand the effect of the quadratic terms in that scenario, this will be the purpose of the next section.

3.3 Inflation on the brane

In this section, we shall consider the inflaton to be an additional scalar field living on the positive tension brane ¹. This setup is equivalent to the one first studied in [129], but as we shall see, some contribution of the Weyl tensor will be taken into account. If the energy scale of inflation is well below the brane tension, this system is well described by the low-energy effective theory of section 2.6. However, if the typical energy scale of inflation is comparable to or larger than the brane tension, we may use the formalism of the previous section to get a better insight.

¹This notion of inflation on the brane should not be confused with the notion of brane inflation introduced in the introduction, where the inflaton scalar field was the distance between the branes. In what follows the inflaton scalar field will be considered to live on the brane.

3.3.1 Background

For simplicity, let us assume that the brane is spatially flat in the background (k = 0). The stress-energy tensor of the inflaton is given by:

$$T_{\mu\nu} = \partial_{\mu}\varphi \,\partial_{\nu}\varphi - \left(\frac{1}{2} \left(\partial\varphi\right)^2 + V(\varphi)\right) \,q_{\mu\nu},\tag{3.32}$$

where the inflaton scalar field φ evolves in a potential $V(\varphi)$ which is assumed to satisfy some slow-roll conditions. The exact slow-roll constraints imposed on the potential will be specified later on. For the background, the energy density for the scalar field can be expressed the usual way:

$$\rho = -T_0^0 = V(\varphi_0) + \frac{1}{2}\dot{\varphi}_0^2, \qquad (3.33)$$

where a dot represents derivative with respect to the proper time and φ_0 is the background value of the scalar field.

When the kinetic energy of the scalar field is assumed to be negligible compared to its potential energy, $\dot{\varphi}_0^2 \ll V(\varphi_0)$, the modified Friedmann equation (2.28) reads:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 \simeq \frac{2V}{L^2\lambda} \left(1 + \frac{V}{2\lambda}\right) \simeq \text{constant.}$$
 (3.34)

where $\lambda = 6/\kappa L$ is the positive tension brane tension (2.16), and we may recall that for a spatially flat background, with pure AdS geometry, the dark energy term \tilde{C} vanishes. It is worth pointing out that an expansion in ρ/λ is equivalent to an expansion in L^2H^2 , in the low-energy limit $L^2H^2 \ll 1$ we indeed have $2\rho/\lambda \sim L^2H^2$. In what follows we will keep all terms in L^2H^2 so any result will remain valid at high energies, when $L^2H^2 \gg 1$. As mentioned before, in the limit where the length scale L is negligible compared to the other length scales of the theory, we should recover the standard four-dimensional low-energy results.

Since the inflaton is confined on the four-dimensional brane, the Klein-Gordon equation for the scalar field will remain unaffected by the presence of the fifth dimension:

$$V_{,\varphi}(\varphi_0) = -\ddot{\varphi}_0 - 3\,H\,\dot{\varphi}_0,\tag{3.35}$$

which is a simple consequence of the conservation of energy $D_{\mu}T^{\mu}_{\nu} = 0$. In the slow-roll regime $\ddot{\varphi}_0 \ll H\dot{\varphi}_0$, this results simplifies and one obtains the conventional relation:

$$\dot{\varphi}_0 \simeq -\frac{V_{,\varphi}}{3H}.\tag{3.36}$$

We may now use the formalism of the previous section to consider scalar and tensor perturbations as well as an estimate of non-gaussianity.

3.3.2 Linear Scalar Perturbations

We now focus on the study of linear isotropic perturbations around this conformally flat background. For this, we choose to work in the longitudinal gauge,

$$ds^{2} = a^{2} (-1 + 2\Phi) d\tau^{2} + a^{2} (1 + 2\Psi) d\mathbf{x}^{2}, \qquad (3.37)$$

$$\varphi(t, \mathbf{x}) = \varphi_0(t) + \delta\varphi(t, \mathbf{x}), \tag{3.38}$$

where τ is the conformal time, and in what follows a prime will designate its derivative: $' \equiv \partial_{\tau} = a \partial_t$.

Anisotropic stress

As mentioned previously, in order for the equation of motion to be consistent with the Bianchi identity the Weyl tensor must include a term cancelling the divergence of the quadratic terms in $T_{\mu\nu}$. Using the background equation of motion, the Weyl tensor should satisfy:

$$D_{\mu}E^{\mu}_{\nu} = -\frac{\kappa^2}{4}T^{\beta}_{\alpha}\left(D_{\beta}(T^{\alpha}_{\nu} - \frac{1}{3}T\delta^{\alpha}_{\nu}) - D_{\nu}(T^{\alpha}_{\beta} - \frac{1}{3}T\delta^{\alpha}_{\beta})\right), \qquad (3.39)$$

$$= -\frac{\kappa^2}{4} \left(\begin{array}{c} 0 \\ -\frac{2}{3} \dot{\varphi}_0^2 \left(-\ddot{\varphi}_0 \,\delta\varphi + \dot{\varphi}_0^2 \,\Phi + \dot{\varphi}_0 \,\delta\dot{\varphi} \right)_{,i} \end{array} \right). \tag{3.40}$$

We stress that the time-like component of this divergence vanishes. This is specific to stress-energy tensors coming from a scalar field and will not, as far as we know, be true for a general fluid. This allows us to decompose the Weyl tensor in the simple form:

$$E_{\mu\nu} = \mathcal{E}_{\mu\nu}^{(L)} + E_{\mu\nu}^{TT}, \qquad (3.41)$$

with
$$\mathcal{E}_{\mu\nu}^{(L)} = \begin{pmatrix} 0 & 0 \\ 0 & X_{,ij} - \frac{1}{3}\nabla^2 X \gamma_{ij} \end{pmatrix}, \quad D_{\mu} \mathcal{E}_{\nu}^{(L)\,\mu} = \begin{pmatrix} 0 \\ \frac{2}{3}\nabla^2 X_{,i} \end{pmatrix}, \quad (3.42)$$

where ∇^2 is the spatial Laplacian. This is remarkable since an expression for the longitudinal part of the Weyl tensor has been found without needing to solve a differential equation involving time derivatives. This is especially interesting since no initial conditions need to be specified. The exact expression for X can then be found by comparing (3.42) with (3.40):

$$\nabla^2 X = \frac{\kappa^2}{4} \dot{\varphi}_0^2 \left(-\ddot{\varphi}_0 \,\delta\varphi + \dot{\varphi}_0^2 \,\Phi + \dot{\varphi}_0 \,\delta\dot{\varphi} \right). \tag{3.43}$$

This provides us with a consistent expression for the Weyl tensor (3.42), which can be substituted into the modified Einstein equation (2.19):

$$G_{\mu\nu} = \frac{6}{\lambda L^2} T_{\mu\nu} - \frac{9}{\lambda^2 L^2} \left(T^{\alpha}_{\mu} T_{\alpha\nu} - \frac{1}{3} T T_{\mu\nu} - \frac{1}{2} T_{\alpha\beta} T^{\alpha\beta} q_{\mu\nu} + \frac{1}{6} T^2 q_{\mu\nu} \right) - \mathcal{E}^{(L)}_{\mu\nu}.$$
 (3.44)

A second interesting feature arises from the ansatz (3.42) when the (ij) (with $i \neq j$) component of this equation is considered. This equation directly points out the presence of an effective *anisotropic* stress X at high energies:

$$X = \Psi - \Phi. \tag{3.45}$$

Since only the ρ^2 terms contribute to X, (eq.(3.43) is quartic in the fields so quadratic in the energy density), we notice that at high energies, the ρ^2 terms play the rôle of an effective anisotropic stress.

For linear perturbations, the (0i)-component of the modified Einstein equation (3.44) reads:

$$\delta\varphi = -\frac{2L}{\kappa\,\dot{\varphi}_0\sqrt{1+L^2H^2}}\left(H\,\Phi + \dot{\Psi}\right).\tag{3.46}$$

Using this expression in (3.43) and (3.45), we can express $\dot{\Phi}$ in terms of Φ and $\Psi, \dot{\Psi}, \ddot{\Psi}$. This result can then be used in the (00)-component of the modified Einstein equation to derive the following relation between Φ and Ψ :

$$\Phi = \left(1 - \frac{L^2 a^2 \dot{H}}{1 + L^2 H^2}\right) \Psi = \left(1 - \frac{a^2}{H} \frac{d}{dt} \left(\ln\sqrt{1 + L^2 H^2}\right)\right) \Psi.$$
(3.47)

Note that the first and second time derivative of Ψ cancel exactly, giving a surprisingly simple expression for the anisotropic stress. In the low-energy regime where $L^2H^2 \ll 1$, the anisotropic part cancels out, and the usual result is recovered.

Substituting this relation between Φ and Ψ into the previous expression we had for Φ in terms of $\Phi, \Psi, \dot{\Psi}, \ddot{\Psi}$ finally gives the decoupled second order equation for Ψ :

$$\Psi'' = \nabla^2 \Psi - 2 \left(\frac{a'}{a} - \frac{\varphi_0''}{\varphi_0'} \right) \left(\Psi' + \frac{a'}{a} \Psi \right) + L^2 (H')^2 \frac{2 - L^2 H^2}{1 + L^2 H^2} \Psi + \frac{L^2 H'' H}{1 + L^2 H^2} \Psi - 2 a H' \left(\Psi - \frac{L^2}{1 + L^2 H^2} \left(H^2 \Psi - H \frac{\varphi_0''}{a \varphi_0'} \Psi + \frac{H}{a} \Psi' \right) \right),$$
(3.48)

where we used derivatives with respect to the conformal time in order to simplify the notation and to compare this equation with the usual results. We may point out that the only difference with the usual four-dimensional scalar perturbations will arise from this second-order equation for Ψ . We have indeed already mentioned that the Klein-Gordon equation remains in the usual four-dimensional form $D_{\mu}T^{\mu}_{\nu} = 0$:

$$\delta\varphi'' = \nabla^2 \,\delta\varphi - 2a^2 \Phi V_{,\varphi} + a^2 \delta\varphi V_{,\varphi\varphi} - \varphi'_0 \left(3\Psi' + \Phi'\right) - 2\frac{a'}{a} \,\delta\varphi'. \tag{3.49}$$

Mukhanov equation

In order to solve the differential equation (3.48), it is useful to perform a change of variables. For that we will work instead in terms of the gauge invariant variable u which is related to the metric perturbation by $u = z\Psi$ with $z = \frac{a}{\varphi'_0 \sqrt{1+L^2H^2}}$. We may point out that its expression in terms of the scalar field perturbations is simply:

$$\delta\varphi = \frac{-2L}{a\,\kappa} \left(u' + \frac{\varphi_0''}{\varphi_0'} u \right),\tag{3.50}$$

ie. precisely the conventional relation between the Mukhanov variable u and the scalar field perturbations [135], after defining the four-dimensional Planck mass to be $M_4^2 = L/\kappa$. Substituting, the Mukhanov variable u into the equation (3.48) for Ψ , one obtain the usual second order differential equation:

$$u_k'' + \left(k^2 - \frac{\beta}{\tau^2}\right)u_k = 0, \qquad (3.51)$$

with
$$\frac{\beta}{\tau^2} \equiv \frac{\theta''}{\theta}, \quad \theta = \frac{H}{\varphi_0'},$$
 (3.52)

which is precisely the standard Mukhanov equation. This is a very critical point: Although the background behaviour is different, the evolution of the perturbations remains the same as in standard four-dimensional gravity. In particular, the braneworld nature of the theory has only altered the relation between the Hubble parameter and the potential. This modification might affect observable quantities, such as the power spectrum of perturbations, but can probably be interpreted as a change of background variables.

We may now follow the usual four-dimensional formalism for inflation [10, 135, 136]. In particular, we will find relevant to follow some of the prescription of [16]. Assuming that β may be treated as a constant – the conditions necessary for this assumption will be specified by the slow-roll parameters later on – an analytical solution of (3.52) can be computed. After decomposing the Mukhanov variable into different k-modes u_k , the integration constant for each one of these modes may be fixed by requiring the fluctuations to be in the Minkowski vacuum well inside the Hubble radius, *ie.* when $k^2\tau^2 \gg \beta$, the modes should be of the form:

$$u_k \sim \frac{ie^{-ik\tau}}{(2k)^{3/2}},$$
(3.53)

$$\delta \varphi_k \sim -\frac{L}{a\kappa} \frac{e^{-ik\tau}}{\sqrt{2k}} \left(1 - \frac{i}{k\tau}\right),$$
(3.54)

which corresponds to the Bunch-Davis vacuum [137]. Imposing this condition, uniquely fixes the free parameters. We then study these modes in the regime which interests us ie.
at long wavelengths, when $k^2 \tau^2 \ll 1$, their expression is slightly more complicated:

$$u_{k} \approx \frac{\sqrt{\pi} k^{-3/2}}{2^{3/2} \sin(n\pi) \Gamma(1-n)} \left(\frac{-k\tau}{2}\right)^{\frac{1}{2}-n} \left[1 - e^{-i\pi n} \frac{\Gamma(1-n)}{\Gamma(1+n)} \left(\frac{-k\tau}{2}\right)^{2n} - \frac{\Gamma(1-n)}{\Gamma(2-n)} \left(\frac{-k\tau}{2}\right)^{2}\right],$$
(3.55)

with the index $n = \sqrt{\beta + \frac{1}{4}}$.

Curvature Perturbation and Power Spectrum

In order to have a physical interpretation of this result, we can relate it to the gauge invariant curvature perturbation on *comoving* hypersurfaces $\zeta_{\varphi} = \Psi - \frac{a'}{a} \frac{\delta \varphi}{\varphi'}$. To be entirely rigorous, one could also consider the curvature perturbations on uniform-energy-density hypersurfaces ζ_{ρ} or the curvature perturbations on uniform-*effective*-energy-density hypersurfaces $\zeta_{\rho_{eff}}$:

$$\zeta_{\varphi} = \Psi - \frac{a'}{a} \frac{\delta\varphi}{\varphi'}, \quad \zeta_{\rho} = \Psi - \frac{a'}{a} \frac{\delta\rho}{\rho'} \quad \text{or} \quad \zeta_{\rho_{eff}} = \Psi - \frac{a'}{a} \frac{\delta\rho_{\text{eff}}}{\rho'_{\text{eff}}}, \tag{3.56}$$

where $T_{\mu\nu}^{\text{eff}}$ is given by the right hand side of (3.44). All those three quantities are conserved at long wavelengths by conservation of energy. We may indeed check this result directly:

$$\zeta_{\rho} = \Psi + \frac{a^{2} \, \delta \rho_{\varphi}}{3 \varphi_{0}^{\prime 2}},
\zeta_{\rho}^{\prime} = -\frac{4 \frac{a^{\prime}}{a} + 2 \frac{\varphi_{0}^{\prime \prime}}{\varphi_{0}^{\prime}}}{3 H^{\prime}} \nabla^{2} \Psi.$$
(3.57)

Since $T_{\mu\nu}^{\text{eff}}$ is also conserved (despite not being linearly related to $T_{\mu\nu}$), we may again argue that $\zeta_{\rho_{eff}}$ is conserved at long wavelengths. If the adiabaticity condition holds for linear perturbations, then $\zeta_{\rho_{eff}}$, ζ_{ρ} and ζ_{φ} should coincide at long wavelengths,

for
$$k^2 \tau^2 \ll 1$$
, $\langle \zeta_{\rho}^2 \rangle \simeq \langle \zeta_{\rho_{eff}}^2 \rangle \simeq \langle \zeta_{\varphi}^2 \rangle$.

In the following we will concentrate on ζ_{φ} , in particular, we derive its behaviour at long wavelengths, in the slow-roll regime:

$$\delta\varphi_k \simeq -\frac{2LH}{\kappa} u \simeq -\frac{iLH}{\sqrt{2\kappa k^{3/2}}} e^{-ik\tau}, \qquad (3.58)$$

$$\zeta_{\varphi} \simeq -\frac{a'}{a} \frac{\delta \varphi_k}{\varphi'_0} \sim \frac{ia}{\sqrt{2} k^{3/2}} \frac{H^2}{\varphi'_0} e^{-ik\tau}.$$
(3.59)

The power spectrum $\mathcal{P} \sim k^3 \langle \zeta^2 \rangle$ is hence given by the standard expression:

٢

$$\mathcal{P} \sim \left. \frac{L^2}{2\kappa^2} \frac{a^2 H^4}{{\varphi'_0}^2} \right|_{\tau=\tau^*} \sim \left. \frac{H^6}{V_{,\varphi}^2} \right|_{\tau=\tau^*}$$
(3.60)

$$\sim \left. \frac{V^3}{M_4^6 V_{,\varphi}^2} \left. (1 + V/2\lambda)^3 \right|_{\tau = \tau^*},$$
(3.61)

with τ^* the time of horizon crossing when k = aH. Once again the departure from the standard four-dimensional inflation is a direct consequence of the modification of the Friedmann equation (3.34) at the background level. Expressed in terms of the potential, the power spectrum will therefore get an overall factor of $\left(1 + \frac{V}{2\lambda}\right)^3$, as mentioned in [129]. For a given potential, it will hence appear to be redder than for chaotic inflation. The spectral index, on the other hand will remain unaffected. Indeed, for *small* β , (we will study in the following the conditions for β to be small), the spectral index is given by:

$$n_S - 1 = \frac{d \ln \mathcal{P}_{\zeta}}{d \ln k} = -2\beta + 2\beta^2 + \mathcal{O}(\beta^3).$$
(3.62)

Slow-Roll Conditions

We may now define the slow-roll parameters as follows:

$$\epsilon = -\frac{H'}{aH^2} \quad \text{and} \quad \lambda^{(n)} = \frac{d^n \ln \epsilon}{d \ln a^n}.$$
(3.63)

Each parameter $\lambda^{(n)}$ may be treated as a constant as long as $\lambda^{(n+1)} \ll 1$, (with $\lambda^{(0)} = \epsilon$). In terms of those parameters, β takes the *exact* form:

$$\beta = -\frac{a^2 H^2 \tau^2}{4} \left[2\lambda^{(2)} - (2 + \lambda^{(1)}) (2\epsilon + \lambda^{(1)}) - \frac{L^2 H^2}{(1 + L^2 H^2)^2} \epsilon (2 + 6\epsilon + L^2 H^2 (2 + 3\epsilon)) \right].$$
(3.64)

In order to obtain an almost scale invariant power spectrum (3.60), with a small spectral index (3.62), two main assumptions have been made on β . β is indeed required to be small, $\beta \ll 1$, and to be almost constant. For β to be small, the parameters ϵ and $\lambda^{(1)}, \lambda^{(2)}$ need to be small. This will be translated into the slow-roll parameter conditions.

If $\lambda^{(1)}$ and $\lambda^{(2)}$ are small, ϵ can be treated as a constant and the scale factor a hence behaves as $a \simeq -H^{-1}\tau^{\frac{1}{\epsilon-1}}$. The overall coefficient in the expression (3.64) for β is therefore $a^2H^2\tau^2 \sim \tau^{-2\frac{\epsilon}{1-\epsilon}}$. Thus it is consistent to treat β as a constant (as has been done to get the expression (3.55)), if the slow-roll conditions are respected: $\epsilon \ll 1$, $\lambda^{(1)} \ll 1$ and $\lambda^{(2)} \ll 1$.

Up to first order in the slow-roll parameters ϵ and $\lambda^{(1)}$, (considering all other $\lambda^{(n)}$ to be negligible), using eq.(3.62), the spectral index takes the form:

$$n_S - 1 = -2\epsilon - \lambda^{(1)} - \frac{L^2 H^2}{1 + L^2 H^2} \epsilon.$$
(3.65)

We will hence recover the standard result for the spectral index [10, 135, 136], if we define the second slow-roll parameter η such that:

$$n_S - 1 = -6\epsilon + 2\eta + \mathcal{O}(\epsilon, \eta). \tag{3.66}$$

In that case, the parameters ϵ and η may be expressed in terms of the scalar field as pointed out in [129, 133, 134]:

$$\epsilon = -\frac{\dot{H}}{H^2} = \frac{L}{2\kappa} \frac{1 + V/\lambda}{(1 + V/2\lambda)^2} \frac{V_{,\varphi}^2}{V^2},\tag{3.67}$$

$$\eta = -\frac{\ddot{H}}{2H\dot{H}} + \frac{1 + \frac{1}{2}L^2H^2}{1 + L^2H^2} \epsilon = \frac{L}{\kappa} \frac{1}{1 + V/2\lambda} \frac{V_{,\varphi\varphi}}{V},$$
(3.68)

If we worked at higher order in the slow-roll parameters, there would be some departure from standard four-dimensional gravity in the spectral index. However, once again, this departure could be eliminated by an adequate redefinition of the third slow-roll parameter ξ^2 (as defined for instance in [133,138]). We do not wish to show the calculation explicitly here, but it can be seen that if the second order parameter is defined in our case such that $\xi^2 = \frac{L^2}{\kappa^2} \frac{V_{,\varphi\varphi\varphi}V_{,\varphi}}{V^2} - \frac{6L^2H^2}{1+L^2H^2} \epsilon^2$, the power spectrum to second order in the slow-roll parameters will recover the same form as in the standard four-dimensional chaotic inflation. At this point, from the knowledge of the amplitude of the scalar perturbations at long wavelengths and their scale-dependence, it is not possible to distinguish between a model of standard four-dimensional chaotic inflation with potential $V^{(4d)}$ satisfying the standard slow-roll conditions with parameters ϵ_{4d} , η_{4d} , ξ_{4d} , and a model of brane inflation with a potential V, such that at the beginning of inflation when the first modes exit the horizon,

$$V^{(4d)} \simeq V\left(1+\frac{V}{2\lambda}\right)\left(1+\frac{V}{\lambda}\right),$$
(3.69)

$$V_{,\varphi}^{(4\mathrm{d})} \simeq V_{,\varphi} \left(1 + \frac{V}{\lambda}\right)^{3/2},$$
(3.70)

$$V_{,\varphi\varphi}^{(\mathrm{4d})} \simeq V_{,\varphi\varphi} \left(1 + \frac{V}{\lambda}\right),$$
(3.71)

$$V_{,\varphi\varphi\varphi}^{(4d)} \simeq \left(1 + \frac{V}{\lambda}\right)^{1/2} \left(V_{,\varphi\varphi\varphi} \left(1 + \frac{V}{2\lambda}\right)^2 + \frac{3}{2} \frac{V_{,\varphi}^3}{\lambda V} \left(1 + \frac{V}{2\lambda}\right)^{-1}\right).$$
(3.72)

We argue that observations of long wavelength scalar perturbations alone are not enough to differentiate between standard chaotic inflation and inflation on a brane with a potential satisfying the modified slow-roll conditions $\epsilon, \eta \ll 1, \eta$ as given in (3.68). Such observations are not sufficient to distinguish between inflation occurring in a purely four-dimensional universe and on a brane embedded in a fifth dimension.

An important point to notice however, is the constraint the potential must be submitted to. In conventional inflation, the slow-roll condition (3.67) $\epsilon \ll 1$ imposes the variation on the potential to be small, *ie.* the potential to be nearly flat:

$$\epsilon^{(4d)} = \frac{L}{2\kappa} \left(\frac{V_{,\varphi}^{(4d)}}{V^{(4d)}} \right)^2 \ll 1.$$
(3.73)

The condition imposed on the "brane potential" V, on the other hand are less constraining:

$$\epsilon = \frac{L}{2\kappa} \frac{1 + V/\lambda}{(1 + V/2\lambda)^2} \frac{V_{\varphi}^2}{V^2} \ll 1.$$
(3.74)

This constraint will be satisfied for a flat potential as for conventional inflation: $\frac{V_{\varphi}^2}{V^2} \ll 1$, but could as well be satisfied for a very large potential: $V \gg \lambda$, for which the flat condition would not be required. This opens up the possibility for steep inflation as discussed in [129,131,139] which might facilitate the end of inflation: Inflation could start with a steep potential at high energies $V \gg \lambda$ and would naturally stop when the order of magnitude of the potential $V(\varphi_0)$ reaches the brane tension λ . Observational constraints that have to be imposed on the brane potential have for instance been studied in [140].

To extend this study we will consider, in the next chapter, typical effects that may arise on the brane due to the non-local nature of the theory. We will see how the behaviour of the perturbations may then differ from the standard case. However we first notice that in the limit of large wavelengths, comparing scalar and tensor perturbations will give a different signature in steep brane inflation than in standard four-dimensional inflation with one scalar field.

3.3.3 Tensor Modes

The scalar field φ is the only source of matter of the theory. The effect of the quadratic term on the behaviour of tensor perturbations will hence strictly be a "background effect" (*ie.* equivalent to introducing some $T^0_{\mu\alpha}T^0_{\nu\beta}h^{(t)\,\alpha\beta}$ -kind of source terms, with $T^0_{\mu\alpha}$ the background value of the stress-energy tensor). In particular, for purely tensor perturbations, the vector part (3.28) of the Weyl tensor must vanish, which will make the derivation very similar to the usual four-dimensional one. Indeed, we can consider the metric perturbation:

$$ds^{2} = -a^{2} d\tau^{2} + a^{2} (\delta_{ij} + h_{ij}) dx^{i} dx^{j}, \qquad (3.75)$$

where the three-dimensional tensor h_{ij} is transverse and traceless $h_i^i = 0$, $\partial_i h_j^i = 0$, and indices are raised with δ^{ij} . With respect to this metric, we may indeed check that $D_{\mu}E_{\nu}^{\mu} = 0$. Using the background equation of motion, the tensor modes satisfy the standard equation:

$$h_{ij}'' + 2\frac{a'}{a}h_{ij}' - \nabla^2 h_{ij} = 0, \qquad (3.76)$$

where the only difference to the chaotic inflation case arises in the relation of the scale factor a to the potential through the modified Friedmann equation (3.33, 3.34). We can therefore treat the tensor perturbations the same way as it is usually done in standard inflation. The power spectrum is given by [129, 136]:

$$\mathcal{P}_g = \frac{72}{M_4^2} H^2|_{\tau=\tau^*} \tag{3.77}$$

$$\simeq \left. \frac{24}{M_4^4} V\left(1 + \frac{V}{2\lambda}\right) \right|_{\tau = \tau^*}.$$
(3.78)

Here again, we notice that for a given potential, the power spectrum for gravitational waves will appear to be redder. However the overall factor is less important than for the scalar power spectrum this will hence be reflected in the ratio.

The tensor spectral index can be derived the usual way:

$$n_T = \frac{d\ln \mathcal{P}_g}{d\ln k} = -\frac{L^2\lambda}{6} \frac{V_{,\varphi}^2}{V^2} \frac{1+\frac{V}{\lambda}}{(1+\frac{V}{2\lambda})^2} = -2\epsilon, \qquad (3.79)$$

and the ratio r of the amplitude of the tensor perturbations at Hubble crossing to the scalar perturbations is modified by a factor $(1 + V/\lambda)^{-1}$:

$$r \simeq \left. \frac{\epsilon}{1 + V/\lambda} \right|_{\tau = \tau^*}.$$
 (3.80)

As mentioned in [129], even though the tensor spectral tilt remains the same as in standard inflation: $n_T = -2\epsilon$, for high energies $V/\lambda \gg 1$, the ratio of tensor to scalar perturbations on the brane will in general be smaller than what is expected from ordinary single scalar field inflation. However for slightly more complicated models of four-dimensional inflation, such as hybrid inflation, the ratio r will similarly be reduced. The relation of tensor to scalar perturbations is thus still not enough to distinguish between such a model and a scenario of brane inflation. The Cyclic-model predicts as well a low amplitude for the tensor modes. In the following chapter we shall hence study some more fundamental features of the five-dimensional nature of this model, but first we might check what the behaviour of the non-linear terms is expected to be.

3.3.4 Estimating the Non-Gaussianity

In order to have a consistency check on the validity of the perturbative approach, we estimate the non-gaussianity corrections to the power spectrum. We consider the nongaussian part to be strongly dominated by the scalar field perturbations. In general, an estimate can be given by comparing the cubic terms in the Lagrangian to the quadratic ones. Unfortunately this method cannot be used here, so as a first approximation we will compare the quadratic term in the redefined gauge invariant comoving curvature to the linear one. Using the result of [141, 142], to second order, the gauge invariant curvature perturbation on uniform-energy-density hypersurfaces $\zeta_{\varphi}^{(2)}$ is given by:

$$\zeta_{\varphi}^{(2)} = \frac{\left(\Psi' + 2\frac{a'}{a}\Psi + \frac{a'}{a}\frac{\delta\varphi'}{\varphi'}\right)^2}{\frac{a''}{a} + \frac{a'}{a^2} - \frac{a'}{a}\frac{\varphi''}{\varphi'}}.$$
(3.81)

Again we could be interested in the curvature perturbation on uniform-effective-energydensity hypersurfaces or on comoving hypersurfaces instead, but for the purpose of this estimate, those three quantities will give a coinciding result. The ratio of the second term to the first one in this expansion gives an estimate which is the same as in the context of slow-roll inflation:

$$\frac{\sqrt{\zeta_{\varphi}^{(2)}}}{\zeta_{\varphi}^{(1)}} \simeq \frac{3}{\sqrt{2}} \epsilon. \tag{3.82}$$

To lowest order in the expansion, the correlation function for the scalar field perturbations is: $\langle \varphi \varphi \rangle \sim \left(\zeta_{\rho}^{(1)} \right)^2$, whereas the cubic terms are: $\langle \varphi \varphi \varphi \rangle \sim \zeta_{\rho}^{(1)} \zeta_{\rho}^{(2)}$ so it does make sense to consider the ratio of the cubic terms to the quadratic ones $\left(\frac{\langle \varphi \varphi \varphi \rangle^2}{\langle \varphi \varphi \rangle^3} \right)^{1/4} \sim \frac{\sqrt{\zeta_{\rho}^{(2)}}}{\zeta_{\rho}^{(1)}} \sim \epsilon$ to estimate the order of magnitude of the non-gaussian terms.

In this approximation, some explicit L^2H^2 corrections will arise at second order in the slow-roll parameter ϵ . However this remains an estimate which aim was only to check that the non-linear terms were indeed damped in comparison to the linear ones. The damping occurs with an order of magnitude proportional to the slow-roll parameter, which is precisely what is found in the context of standard four-dimensional inflation [141].

3.4 Discussion

In the first part of this chapter, we derived a consistent formalism for the two-brane RS model capable of modelling the quadratic terms in the stress-energy in a covariant way.

Although this formalism could be complicated in general, it is completely generalisable to any other braneworld model. We have, for instance explained how the different ideas could be used to study the asymmetric terms present on a general bulk brane not submitted to any \mathbb{Z}_2 symmetry.

This work relies on the important assumption that the branes remain conformal to each other. Although this assumption is not true when KK corrections are taken into account; we will see in the next chapter that this work may be extended in order to accommodate this fact. But starting with this assumption, we have seen in this chapter that much analytical progress could be made.

Another important assumption has been made in this chapter. After having separated out in the Weyl tensor a part $E_{\mu\nu}^{TT}$ which is transverse, traceless and vanishes for the background, we have argued that for the purpose of long wavelength adiabatic perturbations, the contribution of $E_{\mu\nu}^{TT}$ could be neglected.

Using these assumptions, an effective theory could be derived for the two-brane RS model, in a regime where the quadratic terms in the stress-energy tensor could play an important rôle. We have then taken the one-brane limit of this theory, considering the background geometry to be purely AdS and bringing the negative tension brane to infinity, leaving the fifth-dimension uncompactified. In that limit the effective theory simplifies considerably, since no backreaction from the negative tension brane is observed on the positive one.

It is in this one-brane limit of the model that we have then studied the effect of the quadratic terms in the stress-energy to brane inflation. For this we have considered the inflaton scalar field to live on the brane (instead of being represented by a moduli), and have solved the modified Einstein equation for brane inflation driven by a scalar field in a slow-roll potential. We showed that the perturbations are anisotropic in longitudinal gauge, in contrast to the case of standard inflation. When the potential for the inflaton is important, or equivalently, when $LH \gg 1$, we showed how to generalise the expression of the standard inflation variables and parameters. The corrections to these terms arise from a purely background effect. Indeed, assuming adiabaticity, the corrections can only influence

CHAPTER 3. QUADRATIC TERMS IN THE STRESS-ENERGY

the background. For a given inflation potential, the power spectrum of both the tensor and the scalar perturbations are redder than in the normal four-dimensional case. Compared to scalar perturbations, the tensor perturbation amplitude is weaker. However for scalar perturbations, this model can be reinterpreted as standard four-dimensional inflation with a redefined potential, giving rise to the same astrophysical observations. These results have already been well-understood in numerous previous works. Among them we can point out the work by R. Maarteens et.al. [129] where the possibility of having different slow-roll constraints on the potential has first been pointed out and then used in many following studies [127,131,133,143] extending the model to hybrid inflation [144], or even to stochastic inflation [145] and working out the non-gaussianities [134, 146]. However, all these studies have relied on the assumption that the Weyl tensor could be entirely negligible. Although, we recovered the same results, our study is an important consistency check as it enables us to verify the prescription and to extend it in the next chapter. The relations between the brane inflation variables and the redefined standard inflation variables have been given with precision, and our results are reliable up to second order in the slow-roll parameters. Another important feature of our four-dimensional effective theory is its straightforward extension to more interesting and realistic scenarios where both boundary branes have their own dynamics. This opens up the possibility of studying a large range of braneworld scenarios.

Chapter 4

Typical effects of Kaluza Klein corrections



Chaac, Rain God (Maya)

4.1 Introduction

In the previous chapter, the analysis of the quadratic term in the stress-energy relied on two main assumptions. The first one was the hypothesis that the two boundary branes metrics could be treated as being conformal. The second one was relying on the idea that the homogeneous part of the Weyl tensor could be neglected for the purpose of adiabatic perturbations. In this chapter we wish to extend this study and in particular to understand the effect of some Kaluza Klein modes which would break both these assumptions. We shall indeed see that the contribution of the KK modes is encoded in the homogeneous part of the Weyl tensor. Furthermore, once some KK modes are taken into account, the metric on both boundary branes may not be taken to be conformal anymore. The two main assumptions of the previous chapter will therefore be relaxed.

In order to consider possible corrections that might arise from the homogeneous part of the Weyl tensor, we shall summarise the different properties such a tensor should have. We can then suggest an ansatz and include the contribution from a specific four-dimensional tensor $A_{\mu\nu}$, that we check is consistent with the proprieties of the Weyl tensor, and thus represents a natural candidate to consider. This tensor represents the term that would be derived if some four-dimensional Weyl-squared terms were included in the four-dimensional effective action.

Around static branes, the first order perturbations on the brane metrics have been solved exactly in chapter 2. We have seen that in the presence of an extra-dimension, an infinite tower of KK modes affects the four-dimensional geometry. As long as the extra dimension is of finite size, the modes are discrete. For static branes, we show that including the contribution of $A_{\mu\nu}$ to the effective theory, correctly reproduces the effects of the first KK mode in the case where matter is present only on one-brane. It seems therefore natural to study the effect of the tensor $A_{\mu\nu}$ in more general setups. We then use this idea in two specific examples.

In the first one we extend the brane inflation treatment from the previous chapter and study the effect of the correction term in that model. In particular we concentrate on the study of scalar and tensor perturbations, and point out the characteristic features which distinguish their evolution from a standard scenario. We give as well an estimate of the non-gaussianity and stress its different contribution.

As a second example, we compare brane inflation and fast-roll models and analyse whether they can be distinguished in terms of these corrections. As mentioned in the introduction of this thesis, the Ekpyrotic and Cyclic models [86–88,147,148] have recently been studied as a potential alternative to inflation, giving a new picture to solve the

4.2. MODEL FOR THE FIRST KK MODE IN THE LOW-ENERGY EFFECTIVE THEORY

homogeneity, isotropy and flatness problems. Some work suggested that the Ekpyrotic and Cyclic models might provide an alternative scenario for the production of a nearly scale-invariant spectrum. While the production of such a spectrum in inflation is based on "slow-roll" conditions in an expanding universe, the Ekpyrotic and Cyclic models, on the other hand, use a "fast-roll" potential in a contracting universe for which the issue of the "beginning of our Universe" is avoided. Further studies [16, 17] have shown an exact duality between the two models in the production of density perturbations making the two models hard to distinguish without bringing in results from the observation of tensor perturbations. In this chapter, we use our prescription to compare the behaviour of the fast-roll and slow-roll models in the case when $A_{\mu\nu}$ contributes to the transverse part of Weyl tensor. We examine how these corrections influence the production of a scaleinvariant spectrum, showing how they enable us to distinguish between general models with slow-roll and fast-roll conditions.

4.2 Model for the First KK Mode in the Low-energy Effective Theory

4.2.1 Modification of the Low-energy Effective Theory

In this section, we suggest a possible extension of the four-dimensional effective theory that is capable of recovering the first KK mode as described in the quasi-static limit of section 2.7.

We work in a low-energy limit, where the density of the matter confined on the branes is much smaller than the brane tensions. In that case, we have seen in section 2.7, that the zero mode of the two brane RS model is described by the four-dimensional effective low-energy theory derived in section 2.6.

Properties of the Weyl tensor

In this theory, the metrics on both branes are conformally related $q_{\mu\nu}^{(-)} = \Psi^2 q_{\mu\nu}^{(+)}$, as it has already been mentioned in the previous chapter (3.1). In the low-energy limit, the modified Einstein equation on the positive tension brane is:

$$G_{\mu\nu}^{(+)} = \frac{\kappa}{L} T_{\mu\nu} - E_{\mu\nu}^{(+)}, \qquad (4.1)$$

with $T_{\mu\nu}$ the stress-tensor of matter on the positive brane. If we introduce matter on this brane only, the expression for the tensor $E_{\mu\nu}^{(+)}$ in this effective theory is (2.46)

$$E_{\mu\nu}^{(+)} = -4\partial_{\mu}\Psi\partial_{\nu}\Psi + 2\Psi D_{\mu}D_{\nu}\Psi - \left(2\Psi\Box\Psi - (\partial\Psi)^{2}\right)q_{\mu\nu}^{(+)} - \Psi^{2}G_{\mu\nu}^{(+)}, \qquad (4.2)$$

where all covariant derivatives are taken with respect to $q_{\mu\nu}^{(+)}$ (as will be the case throughout this section unless otherwise specified). Furthermore the low-energy effective theory predicts the conformal relation (2.42) between the induced Weyl tensor on each brane: $E_{\mu\nu}^{(-)} = \Psi^{-2} E_{\mu\nu}^{(+)}$. These relations (4.2, 2.42) for $E_{\mu\nu}^{(\pm)}$ can be checked to model correctly the behaviour of the zero mode (*ie.* the long wavelength limit of the exact five-dimensional theory) and the conformal relation (2.42) can be checked for the background. However there is no reason for expressions (4.2, 2.42) to remain exact for non-conformally flat spacetimes in general. The electric part of the Weyl tensor $E_{\mu\nu}^{(\pm)}$ is indeed the only quantity which remains unknown from a purely four-dimensional point of view as it encodes information from the bulk geometry. It is through this term that the bulk generates KK corrections on the brane.

We have seen in chapter 2, that from the properties of the five-dimensional Weyl tensor (2.4), $E_{\mu\nu}^{(\pm)}$ is traceless. Furthermore, in the low-energy limit, by the Bianchi identity, $E_{\mu\nu}^{(\pm)}$ is divergenceless. Outside the low-energy limit, as was the case in the previous chapter, $E_{\mu\nu}$ was not transverse, but we have seen in section 3.2.1, that a transverse and traceless part noted as $E_{\mu\nu}^{TT}$ could be separated out in the expression for $E_{\mu\nu}^{(+)}$. What will be said for $E_{\mu\nu}$ in what follows in the low-energy limit, can therefore equivalently be applied to $E_{\mu\nu}^{TT}$ when the low-energy constraint is relaxed.

Possible modifications from the Weyl tensor

If we want to modify the four-dimensional effective theory of section 2.6, the only modification that would be consistent with the five-dimensional nature of the theory is to add a correction to the expression (4.2) of $E_{\mu\nu}^{(+)}$, (or to (2.42) for $E_{\mu\nu}^{(-)}$). Motivated by [149, 150], let us consider a possible term $E_{\mu\nu}^{\text{corr}}$ that could be added as a correction to (4.2). We shall now summarise the properties $E_{\mu\nu}^{\text{corr}}$ should have:

- In the low-energy limit, the relation (4.2) is exact for the background. $E_{\mu\nu}^{\text{corr}}$ therefore needs to vanish for conformally flat spacetimes. We shall see in the next chapter that the expression (4.2) for $E_{\mu\nu}^{(+)}$ does not model correctly higher order in velocities. But beyond the low-energy regime, we will instead consider corrections to $E_{\mu\nu}^{TT}$, which does vanish for the background even at high energies. Thus the corrections to that tensor need to vanish for the background as well.
- Since $E_{\mu\nu}^{(+)}$ is transverse, so is $E_{\mu\nu}^{\text{corr}}$, and we might think of it as being derived from an action S_E . It is however important to point out that the existence of such an action cannot in general be proofed. The requirement that $E_{\mu\nu}^{\text{corr}}$ derives from an action is hence not always valid, but $E_{\mu\nu}^{\text{corr}}$ still needs to be conserved.
- Furthermore, since $E_{\mu\nu}^{\text{corr}}$ is traceless, the action S_E (assuming there is one) must be conformally invariant. Indeed, the variation of this action under a conformal transformation with $\delta q_{\mu\nu} \propto q_{\mu\nu}$ will be:

$$\delta S_E = \frac{1}{2} \int \mathrm{d}^4 x \sqrt{-q} \, E^{\mathrm{corr}}_{\mu\nu} \delta q^{\mu\nu}, \qquad (4.3)$$

and since $E_{\mu\nu}^{\text{corr}}$ is traceless, the action S_E must be conformally invariant.

• Finally, if we want to modify the effective theory in order to accommodate the first KK corrections present in (2.73), we need to add to the action a term of fourth order in derivatives.

If we want to consider a contribution $E_{\mu\nu}^{\text{corr}}$ arising from the Weyl tensor, which might be able to model the first KK mode in some limit, this correction term $E_{\mu\nu}^{\text{corr}}$ has to satisfy the previous properties. Furthermore, for simplicity, we will restrict ourselves to a correction $E_{\mu\nu}^{\text{corr}}$, which is a functional of the metric only. It could be argued that there is a priori no reason why this term should be expressed locally in terms of four-dimensional quantities, however this will be our ansatz, as motivated by [149, 150]. Then the most straightforward term derived from a conformally invariant action, which vanishes for conformally flat spacetimes and is of higher order in derivative, is [151]:

$$A_{\mu\nu}[q] = \frac{1}{2\sqrt{-q}} \frac{\delta S_{C^2}}{\delta q^{\mu\nu}} \tag{4.4}$$

with the action S_{C^2} defined as:

$$S_{C^2} = \int \mathrm{d}^4 x \sqrt{-q} \ C_{\alpha\beta\gamma\delta} C^{\alpha\beta\gamma\delta}. \tag{4.5}$$

The stress-tensor derived from the action is then:

$$A_{\mu\nu}[q] = -\Box R_{\mu\nu} + \frac{1}{3} D_{\mu} D_{\nu} R + \frac{1}{6} \Box R q_{\mu\nu} - \frac{1}{6} R^2 q_{\mu\nu}$$

$$+ \frac{2}{3} R R_{\mu\nu} + \frac{1}{2} R_{\alpha\beta} R^{\alpha\beta} q_{\mu\nu} - 2 R_{\mu\alpha\nu\beta} R^{\alpha\beta}.$$
(4.6)

Here all covariant derivatives are taken with respect to the general metric $q_{\mu\nu}$. Since the Weyl tensor vanishes for conformally flat spacetimes, the tensor $A_{\mu\nu}[q]$ has all the requirements it should satisfy: It is indeed traceless, transverse and vanishes for the background. Adding a term $E_{\mu\nu}^{\text{corr}} \propto A_{\mu\nu}[q^{(+)}]$ to the expression (4.2) is therefore consistent with the Gauss-Codacci and the Bianchi identities and is consistent with the background results. Following this argument, it seems to us natural to consider the effects that the addition of such terms would have in the theory.

Moreover, from the results of section 2.7, we can point out that when the first KK corrections are taken into account, the brane metrics do not remain conformal to each other any longer. We therefore need to modify the relation (2.37) between the brane metrics. In this procedure we suggest breaking the conformal relation by including some independent contribution to $E_{\mu\nu}^{(\pm)}$. By independent, we mean that the corrections will not satisfy the conformal relation (2.42), *ie.* $E_{\mu\nu}^{(-)\,\text{corr}} \neq \Psi^{-2} E_{\mu\nu}^{(+)\,\text{corr}}$.

4.2.2 Ansatz

Our idea is to include a contribution from the stress-tensor (4.6) in the effective theory. More precisely, we will include a term proportional to $A_{\mu\nu}[q^{(\pm)}]$ to the electric part of the Weyl tensor. We will therefore consider the theory governed by the four-dimensional equations:

$$G_{\mu\nu}^{(+)} = \frac{\kappa}{L} T_{\mu\nu} + 4\partial_{\mu}\Psi \partial_{\nu}\Psi - 2\Psi D_{\mu}D_{\nu}\Psi + \left(2\Psi \Box \Psi - (\partial\Psi)^{2}\right)q_{\mu\nu}^{(+)}$$
(4.7)
+ $\Psi^{2}G_{\mu\nu}^{(+)} - E_{\mu\nu}^{(+)\,\text{corr}}.$

$$G_{\mu\nu}^{(-)} = \frac{1}{\Psi^2} \left(-2\Psi D_{\mu} D_{\nu} \Psi + 4\partial_{\mu} \Psi \partial_{\nu} \Psi + \left(2\Psi \Box \Psi - (\partial \Psi)^2 \right) q_{\mu\nu}^{(+)} \right)$$
(4.8)

$$+ G_{\mu\nu}^{(+)} - E_{\mu\nu}^{(-)\,\text{corr}},$$

$$\Box \Psi = \frac{1}{6} R^{(+)} \Psi,$$
(4.9)

where again, all derivatives are performed with respect to $q_{\mu\nu}^{(+)}$, and with the corrections terms:

$$E_{\mu\nu}^{(+)\,\text{corr}} = \alpha \, L^2 A_{\mu\nu}[q^{(+)}], \qquad (4.10)$$

$$E_{\mu\nu}^{(-)\,\text{corr}} = \beta \, L^2 A_{\mu\nu}[q^{(-)}], \qquad (4.11)$$

We shall now test this modified four-dimensional effective theory against the exact results (2.73) and (2.74) obtained from five-dimensional analysis.

4.2.3 Consistency check of the Ansatz for Static Branes

In this section we shall check the previous ansatz against the exact solution derived in section 2.7, in the case of perturbations around static branes. The ansatz we propose has to be considered as a first order correction and we will check whether it correctly reproduces the behaviour of the first KK mode from that section. We will therefore not solve the system completely but only work out the zero mode and the first order correction generated by the additional terms in (4.10), (4.11). We study the same scenario as in section 2.7 and we consider metric perturbations around static branes, sourced by the

addition of matter on the positive tension brane. In the background, the positive tension brane is located at y = 0 and the negative one at y = d. To first order in perturbations,

$$q_{\mu\nu}^{(+)} = \eta_{\mu\nu} + \bar{h}_{\mu\nu}^{(+)} \tag{4.12}$$

$$\Psi = \Psi_0 + \delta \Psi, \quad \Psi_0 = e^{-d/L} = \text{const}$$
(4.13)

$$q_{\mu\nu}^{(-)} = \Psi_0^2 \ \eta_{\mu\nu} + \bar{h}_{\mu\nu}^{(-)}, \tag{4.14}$$

and we consider matter to be present at the perturbed level only on the positive tension brane: $T_{\mu\nu}^{(+)} = \delta T_{\mu\nu}^{(+)}$. To that order in perturbations, we can consider the matter fields and the perturbed scalar field $\delta \Psi$ to live on the background metric:

$$R^{(+)} = -\frac{\kappa}{L} \,\delta T^{(+)},\tag{4.15}$$

$$\Box \delta \Psi = -\frac{\kappa}{6L} \Psi_0 \,\delta T^{(+)},\tag{4.16}$$

$$A_{\mu\nu}^{(+)} = -\Box R_{\mu\nu}^{(+)} + \frac{1}{3} R_{,\mu\nu}^{(+)} + \frac{1}{6} \Box R^{(+)} \eta_{\mu\nu}, \qquad (4.17)$$

giving rise to the modified Einstein equation for the positive tension brane:

$$R_{\mu\nu}^{(+)} = \frac{\kappa/L}{1-\Psi_0^2} \left[\delta T_{\mu\nu}^{(+)} - \frac{1}{2} \, \delta T^{(+)} \eta_{\mu\nu} + \frac{\Psi_0^2}{6} \, \delta T^{(+)} \eta_{\mu\nu} + \frac{\Psi_0^2}{3\,\square} \, \delta T_{,\mu\nu}^{(+)} \right] \qquad (4.18)$$
$$- \frac{\alpha\kappa/L}{(1-\Psi_0^2)^2} \, L^2 \square \left[\delta T_{\mu\nu}^{(+)} - \frac{1}{3} \, \delta T^{(+)} \eta_{\mu\nu} + \frac{1}{3\,\square} \, \delta T_{,\mu\nu}^{(+)} \right] \\+ \frac{\kappa}{L} \, \mathcal{O} \left(\alpha^2 L^4 \square^2 \delta T_{\mu\nu}^{(+)} \right).$$

To get this result, we have only considered the first order corrections generated by the addition of the term (4.6) in the effective four-dimensional theory. We notice that up to first order in perturbations and to first order in the correction term $\alpha L^2 \square$, we have the remarkable relation:

$$A_{\mu\nu}^{(+)} = -\frac{\kappa/L}{1-\Psi_0^2} \,\Box \Sigma_{\mu\nu},\tag{4.19}$$

with the tensor $\Sigma_{\mu\nu}$ as given in (2.62). This is an important result, based on which, our ansatz will be verified.

For the negative tension brane, the modified Einstein equation reads:

$$R_{\mu\nu}^{(-)} = R_{\mu\nu}^{(+)} - 2\frac{\delta\Psi, \mu\nu}{\Psi_0} - \frac{\Box\delta\Psi}{\Psi_0}\eta_{\mu\nu} - \beta L^2 A_{\mu\nu}^{(-)}, \qquad (4.20)$$

so that:

$$R_{\mu\nu}^{(-)} = \frac{\kappa/L}{1 - \Psi_0^2} \left[1 - \left(\beta (1 - \frac{1}{\Psi_0^2}) + \frac{\alpha}{1 - \Psi_0^2} \right) L^2 \Box \right] \Sigma_{\mu\nu}.$$
 (4.21)

In the de Donder gauge, the Ricci tensor is related to the metric perturbation by:

$$R^{(\pm)}_{\mu\nu} = -\frac{1}{2} \,\Box^{(\pm)} \bar{h}^{(\pm)}_{\mu\nu}. \tag{4.22}$$

The perturbation of the brane metrics is therefore:

$$\bar{h}_{\mu\nu}^{(+)} = -2\frac{\kappa L}{1-\Psi_0^2} \left[\frac{1}{L^2\Box} - \frac{\alpha}{1-\Psi_0^2} + \mathcal{O}(L^2\Box) \right] \Sigma_{\mu\nu} + \frac{2\kappa}{3L\Box^2} \,\delta T_{,\mu\nu}^{(+)} + \frac{\kappa}{3L\Box} \,\delta T^{(+)}\eta_{\mu\nu}, (4.23)$$

$$\bar{h}_{\mu\nu}^{(-)} = -2\frac{\kappa L \Psi_0^2}{1 - \Psi_0^2} \left[\frac{1}{L^2 \Box} - \left(\beta (1 - \frac{1}{\Psi_0^2}) + \frac{\alpha}{1 - \Psi_0^2} \right) \right] \Sigma_{\mu\nu}.$$
(4.24)

We can compare these results with the exact zero and first modes obtained in (2.73) and (2.74). The zero modes agree perfectly as one should expect. This just confirms the well-known result that the effective theory gives the correct zero mode on the brane in the low-energy regime.

More remarkable is the result for the first KK mode. Comparing the expression (4.23) with (2.73), the results agree perfectly if the dimensionless constant α is fixed to the value:

$$\alpha = (1 - \Psi_0^2) f(0) = \frac{2d^3}{3L^3} + \mathcal{O}\left(\frac{d^4}{L^4}\right).$$
(4.25)

Similarly, we have a perfect agreement between the ansatz for the negative tension brane (4.24) and the exact result (2.74) if the constant β is fixed to the value:

$$\beta = \frac{\Psi_0^2}{1 - \Psi_0^2} \left(f(0) - f(d) \right). \tag{4.26}$$

This is an original result. It has been possible to extend the notion of a four-dimensional effective theory in order to accommodate the first KK correction arising from perturbations around static branes.

We should however note that in order to fit to the exact results, the coefficients α and β have to depend on the distance between the branes, or equivalently depend on the scalar field Ψ . If the scalar field is not fixed in the background, $\partial_{\mu}\Psi_{0} \neq 0$, in general the action

 $\int d^4x \sqrt{-q} \,\alpha(\Psi) C_{\alpha\beta\gamma\delta} C^{\alpha\beta\gamma\delta}$ is not conformally invariant, and our procedure is not valid. The generalisation of the effective theory to the case for which the branes are not fixed in the background is therefore not straightforward. In that case a more general analysis is needed as in [152] for the study of perturbations. However, to our knowledge, an exact five-dimensional derivation for the KK modes in a more general scenario has not yet been studied. It is therefore difficult to check any new ansatz for a four-dimensional effective theory in any more elaborate scenario. We hope that this work can be used in order to check the validity of a four-dimensional effective theory and gives insight on how to generalise it to take some typical five-dimensional features into account. When the distance between the branes is taken to be small, as will be the case in the next chapter, it is possible to derive the contribution of KK mode in a more systematic way. In particular in section 5.7, we show the validity of our ansatz in that regime and its possible extension for more general cases, beyond the low-energy approximation. This gives an extra argument to motivate the study of $A_{\mu\nu}$ as a possible correction to the effective theory, modelling part of the first KK mode.

4.3 Effects of the C^2 Terms on Brane Inflation

In section 3.2.2, we have pointed out the presence of a transverse and traceless term in the Weyl tensor $E_{\mu\nu}^{TT}$ which has been neglected in the treatment for brane inflation in section 3.3. This was motivated by the fact that $E_{\mu\nu}^{TT}$ vanishes for the background and we can check that within this approximation, we recover the long wavelength limit of the exact five-dimensional theory. We gave also an argument to justify this approximation in the regime of long wavelength adiabatic perturbations for scalar fields. However, there is no reason for the tensor $E_{\mu\nu}^{TT}$ to remain negligible in general. This term actually encodes information about the bulk geometry. At the perturbed level, the bulk geometry is not purely AdS nor even SAdS anymore. The fluctuations in the bulk geometry will generate some KK corrections on the branes. Those KK modes are mediated by the only term which remains undetermined from a purely four-dimensional point of view: the tensor $E_{\mu\nu}^{TT}$.

In this section, we modify the treatment of section 3.2.2 in order to study the typical kind of corrections that may arise from the perturbed bulk geometry. The only modification consistent with the overall five-dimensional nature of the model is to add a contribution coming from $E_{\mu\nu}^{TT}$. We want to modify the effective theory in order to accommodate terms that are negligible in the long wavelength limit of the five-dimensional theory. We therefore need to consider terms of higher order in derivatives (compared to the other terms already present in the theory). Following the idea of the previous section, a natural ansatz for corrections coming from $E_{\mu\nu}^{TT}$ would be the introduction of the tensor $A_{\mu\nu}$ defined in (4.4) or (4.6). The purpose of this section is therefore to study the effect of the introduction of such terms in the model of brane inflation studied in the previous chapter, section 3.3. The modified Einstein equation for this one-brane RS model is therefore:

$$R_{\mu\nu} = \frac{\kappa}{L} \left(T_{\mu\nu} - \frac{1}{2} T q_{\mu\nu} \right) - \frac{\kappa^2}{4} \left(T^{\alpha}_{\mu} T_{\alpha\nu} - \frac{1}{3} T T_{\mu\nu} \right) - E_{\mu\nu}, \qquad (4.27)$$
$$E_{\mu\nu} = \mathcal{E}^{(L)}_{\mu\nu} + E^{TT}_{\mu\nu}, \qquad (4.28)$$

where $T_{\mu\nu}$, as defined in (3.32), is the stress-energy for the inflaton introduced on the positive tension brane. α is a dimensionless parameter that we assume to be small. All through this study, we will work only up to first order in α . We may emphasise again that $A_{\mu\nu}$ does not alter the background behaviour and will be present only at the perturbed level.

4.3.1 Tensor Perturbations

We consider the previous scenario of brane inflation from section 3.3 and keep all terms in L^2H^2 so that the analysis remains consistent at high energies. To start with, we consider tensor perturbations in detail. We denote by h_{ij} the three-dimensional tensor perturbation. By the same argument used previously, the quadratic terms in the modified Einstein equation have only a "background" effect on the tensor perturbations. In particular, they

can be derived from the action:

$$S_t^{(2)} = \int d^4x \left(a^2 \eta^{\mu\nu} \partial_\mu h^i_j \partial_\nu h^j_i + \alpha L^2 \eta^{\mu\nu} \eta^{\alpha\beta} \left(2 \partial_{\mu\alpha} h^i_j \partial_{\nu\beta} h^j_i - \partial_{\mu\nu} h^i_j \partial_{\alpha\beta} h^j_i \right) \right).$$
(4.29)

This gives rise to the equation of motion for h_{ij} (omitting the *ij*-indices for now on):

$$h'' + 2\frac{a'}{a}h' - \nabla^2 h + \frac{\alpha L^2}{a^2} \left(h'''' - 2\nabla^2 h'' + \nabla^4 h\right) = 0.$$
(4.30)

Where again a prime designates a derivative with respect to the conformal time and ∇^2 represents the three-dimensional spatial Laplacian. We use here the assumption that the constant α is small and expand h as a series in α : $h = h_0 + \alpha h_1 + \mathcal{O}(\alpha^2)$, where $h_{0,1}$ do not depend on α . To lowest order in α , we should recover the results from section 3.3.3. h_0 therefore satisfies: $h''_0 + 2\frac{a'}{a}h'_0 - \nabla^2 h_0 = 0$. Keeping only the terms up to first order in α , the last term in eq.(4.30) can be written as:

$$h'''' - 2\nabla^{2}h'' + \nabla^{4}h = h_{0}'''' - 2\nabla^{2}h_{0}'' + \nabla^{4}h_{0} + \mathcal{O}(\alpha)$$

$$= -2\underbrace{\left[\frac{a'''}{a} - 9\frac{a'a''}{a^{2}} + 12\frac{a'^{3}}{a^{3}}\right]}_{=2aa'H^{2}\epsilon+\epsilon \mathcal{O}(\epsilon,\eta)}h' + 4\underbrace{\left[\frac{2a'^{2}}{a^{2}} - \frac{a''}{a}\right]}_{=a^{2}H^{2}\epsilon} \nabla^{2}h + \mathcal{O}(\alpha).$$

$$(4.31)$$

Using this approximation, equation (4.30) simplifies considerably:

$$h'' + 2\frac{a'}{a} \left(1 - 2\alpha L^2 H^2 \epsilon\right) h' - \left(1 - 4\alpha L^2 H^2 \epsilon\right) \nabla^2 h = \mathcal{O}(\alpha^2).$$
(4.32)

In this result, the terms beyond first order in the slow-roll parameters have been omitted. Eq.(4.32) is consistent with the original eq.(4.30) if we study only the terms up to first order in α in the expression for h. This is an important assumption since the fourth order differential equation reduces to a second order one, allowing us to specify only two parameters on the initial Cauchy surface instead of four. Using this assumption, the requirement that we recover a normalised Minkowski vacuum when the modes are well inside the horizon is then enough to specify h_{ij} . Similarly to the standard case, it is simpler to study instead the evolution of the associated variable u:

$$u = zh, (4.33)$$

$$z = a \left(1 + \alpha L^2 H^2 \left(1 + \epsilon \right) \right), \tag{4.34}$$

so that the second order differential equation for the tensor perturbations simplifies to:

$$u'' + \left(c_h^2 k^2 - \frac{\beta}{\tau^2}\right) u = 0, \tag{4.35}$$

where, up to first order in the slow-roll parameter ϵ and in α , the speed of sound c_h and the parameter β are defined as:

$$c_h^2 = 1 - 4\epsilon \,\alpha L^2 H^2, \tag{4.36}$$

$$\beta = 2 + 3\epsilon \left(1 - 2\alpha L^2 H^2\right). \tag{4.37}$$

The main point to notice is that the tensor modes do not propagate at the speed of light anymore but at the speed of sound c_h . The C^2 -corrections (as introduced in $A_{\mu\nu}$) will therefore modify the effective speed of linear perturbations on the brane. There is a priori no reason for the constant α to be positive. If α is negative, it is possible that the speed of sound will be effectively larger than the speed of light from a four-dimensional point of view. This may result in instabilities at higher order, but we believe that this should be interpreted the same way as if the phase velocity of a fluid is larger than the speed of light. In that case no information is actually transported faster than the speed of light.

The amplitude of the tensor perturbations at sound horizon crossing $(aH = c_h k)$ is multiplied by an extra mode-dependent factor:

$$\mathcal{P}_g \simeq \left. \frac{24}{M_4^4} V\left(1 + \frac{V}{2\lambda}\right) \left(1 - 2\alpha L^2 H^2\right) \right|_{\tau = \tau^*}.$$
(4.38)

The effect of the C^2 -corrections is extremely simple and the extra term becomes negligible at low-energy and at long wavelengths $k^2\tau^2 \ll 1$, but it is important to point out that this is not negligible at high-energy. Similarly, the tensor spectral index is modified by an extra mode-dependent factor:

$$n_T = -2\epsilon \left(1 - 2\alpha L^2 H^2\right). \tag{4.39}$$

Neglecting the slow-roll parameters, H may be treated as constant in this result. We can interpret the previous result as a background redefinition:

$$\bar{\epsilon} = \epsilon \left(1 - 2\alpha L^2 H^2 \right), \tag{4.40}$$

even if the C^2 -corrections did not perturb the background behaviour.

4.3.2 Scalar Perturbations

The scalar perturbations can be treated within the same philosophy as the tensor perturbations so some of the details will be skipped.

In longitudinal gauge, for scalar isotropic perturbations, the contribution of the tensor $A_{\mu\nu}$ is:

$$A_{\mu\nu} = \frac{2}{3a^2} \begin{bmatrix} \nabla^4 & \nabla^2 \partial_i \\ \nabla^2 \partial_j & \left[\frac{1}{2} \left(\partial_\tau^2 - \nabla^2 \right) \left(\partial_{ij} - \delta_{ij} \nabla^2 \right) + \partial_{ij} \partial_\tau^2 \right] \end{bmatrix} (\Psi + \Phi).$$
(4.41)

Introducing this term in the modified Einstein equation (4.27), the equation (3.52) for u gets modified to:

$$u = \tilde{z}\Psi,\tag{4.42}$$

$$u'' + \left(c_u^2 k^2 - \frac{\tilde{\beta}}{\tau^2}\right) u = 0, \qquad (4.43)$$

with the modified parameters:

$$\tilde{z} = z \left(1 + \alpha \, z_1 \right), \tag{4.44}$$

$$z_1 = \frac{4L^2H^2}{3} \left(1 + \frac{1}{2} \frac{L^2H^2}{1 + L^2H^2} \epsilon \right) + 2L^2H^2 \left(1 + 2\eta - \frac{1 - \frac{1}{2}L^2H^2}{1 + L^2H^2} \epsilon \right),$$
(4.45)

$$c_u^2 = 1 + 8\alpha L^2 H^2 \left(1 + \frac{2}{3} \frac{1 + 7/4L^2 H^2}{1 + L^2 H^2} \epsilon \right), \qquad (4.46)$$

$$\tilde{\beta} = \beta + 4\alpha L^2 H^2 \epsilon. \tag{4.47}$$

In the next section, the exact result is derived in the low-energy regime. In particular it is shown that the duality between slow-roll and fast-roll conditions in the production of a scale invariant spectrum remains valid when those kinds of corrections are taken into account. But for now, for simplicity, we consider the first order terms in the slow-roll parameter only; in the previous result, terms beyond first order in α or in the slow-roll parameters have been omitted.

We can notice here the same phenomenon as for the tensor perturbations: the scalar perturbations do not propagate at the speed of light any more but at the speed of sound c_u , which is not the same speed as for tensor perturbations.

The rest of the discussion remains the same. Assuming again that $\tilde{\beta}$ may be treated as a constant, an analytical solution can be found for (4.43) and the constants are chosen so that we obtain Minkowski vacuum $u_k \approx \frac{i e^{-ic_u k\tau}}{(2 c_u k)^{3/2}}$ in the regime where $c_u^2 k^2 \tau^2 \gg \tilde{\beta}$. The expression between the scalar field perturbations $\delta \varphi$ and the variable u is:

$$\delta\varphi = -\frac{2L}{a\kappa} \left[\left(1 + 2\alpha L^2 H^2 + \mathcal{O}(\epsilon) \right) u' + \frac{\varphi_0''}{\varphi_0'} \left(1 + 2\alpha L^2 (2\nabla^2 + H^2 + \mathcal{O}(\epsilon)) \right) u \right], \quad (4.48)$$

so that at short distances, the spacetime is locally flat and we recover the Bunch-Davis vacuum for the scalar field perturbations:

for
$$c_u^2 k^2 \tau^2 \gg \beta^2$$
,
 $u_k \approx \frac{ie^{-ic_u k\tau}}{(2c_u k)^{3/2}}$,
 $\delta \varphi_k \approx -\frac{L}{a\kappa} \frac{e^{-ic_\varphi k\tau}}{\sqrt{2c_\varphi k}} \left(1 - \frac{i}{c_\varphi k\tau}\right) \left(1 + 6\alpha L^2 H^2\right)$, (4.49)
with $c_u = 1 + 4\alpha L^2 H^2 + \mathcal{O}(\epsilon)$ and $c_\varphi = 1 + 8\alpha L^2 H^2 + \mathcal{O}(\epsilon)$.

We point out that the scalar field perturbations propagate at a speed of sound slightly different from the speed of sound of the Mukhanov scalar u.

Following the same procedure as before, the spectral index for scalar perturbations reads:

$$n_S - 1 = -6\epsilon + 2\eta - 8\alpha L^2 H^2 \epsilon, \qquad (4.50)$$

which could be again interpreted as a redefinition of the slow roll-parameter:

$$\tilde{\eta} = \eta + 10\alpha L^2 H^2 \epsilon. \tag{4.51}$$

The overall amplitude of the scalar perturbations gets a mode-dependent factor:

$$\mathcal{P} \sim \left. \frac{V^3 \left(1 + \frac{V}{2\lambda} \right)^3}{M_4^6 V_{,\varphi}^2} \left(1 - 16 \,\alpha \, L^2 H^2 \right) \right|_{\tau = \tau^*/c_u}.$$
(4.52)

We obtain the same kind of corrections as for tensor perturbations (4.38) which again might be significant at high energies. Since the scalar perturbations propagate at a speed of sound c_u , there is an extra factor when evaluating (4.52) due to the fact that the modes exit the horizon at a different time: $V\left(1+\frac{V}{2\lambda}\right)\Big|_{\tau^*/c_u} \sim V\left(1+\frac{V}{2\lambda}\right)\left(1-8\epsilon\alpha L^2H^2\right)\Big|_{\tau^*}$, but this difference is negligible at long wavelengths.

The ratio between the amplitude of the tensor perturbations to the scalar ones acquires the additional mode-dependent factor (to first order in ϵ):

$$r = \frac{\epsilon}{1 + V/\lambda} \left(1 + 16\alpha L^2 H^2 \right) \Big|_{\tau \approx \tau^*}.$$
(4.53)

We may also check that non-linear corrections remain small in the context of this perturbative approach:

$$\frac{\sqrt{\zeta_{\rho}^{(2)}}}{\zeta_{\rho}^{(1)}} \simeq \frac{3}{\sqrt{2}} \epsilon - 8\sqrt{2}\alpha L^2 H^2 \bigg|_{\tau=\tau^*}.$$
(4.54)

We can see that the ratio is slightly modified by a mode-dependent term which is *not* damped by the slow-roll parameter. Within this approach the correction term is required to be small since we are making an expansion in α . However, a region can exist for which α might be small but still important compared to ϵ . In this case some deviations from non-gaussianity might be observed. (We can point out that they will not be present at low energies, when the terms L^2H^2 are negligible.) This is one of the most interesting consequences of the addition of the C^2 -corrections within the context of brane inflation. Another result is presented in the next section, where we will study the consequences of those corrections while considering the production of perturbation with a fast-roll potential.

4.3.3 Slow-roll versus Fast-roll

It has been shown [16, 17] that within the low-energy approximation, there is an exact duality between "inflation"-like and "Ekpyrotic or Cyclic"-like potentials which give rise to the same observational features (for scalar perturbations). We intend to study here whether this duality is preserved when the C^2 -corrections are taken into account. In the setup of the previous section the scalar field was confined on the brane. We will make the analogy between this case and the case for which the scalar field can be interpreted as the dilaton in a two-brane RS model. In that case the dilaton does not live on a specific brane, however from the four-dimensional effective point of view, the formalism will be the same. In what follows we will work in a low-energy regime and neglect the quadratic terms in the stress-energy tensor. In the Cyclic scenario, the two boundary branes from the RS model are taken to be empty (at the time when the fluctuations responsible for the structure are produced) and the effective four-dimensional theory as derived in section 2.6 is modified by the introduction of a potential V:

$$S = \frac{L}{\kappa} \int d^4x \sqrt{-q} \left(\frac{1}{2}R - \frac{1}{2} \left(\partial \phi \right)^2 - V(\phi) \right).$$
(4.55)

V represents an interaction between the two branes which may come from bulk fields such as in the Goldberger-Wise mechanism. The metrics on the branes are conformally related to the effective one by the relations (2.51). The action (4.55) gives some equations of motion which are consistent with the background ones. However the addition of the term $\alpha L^2 C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}$ in this action will not alter the background behaviour and thus will be a consistent term to consider. When deriving the covariant four-dimensional effective action in section 2.6, there is indeed no reason why such a term could not have been introduced. We will therefore consider the following action:

$$S = \frac{L}{\kappa} \int d^4x \sqrt{-q} \left(\frac{1}{2} R - \frac{1}{2} (\partial \phi)^2 - V(\phi) + \alpha L^2 C_{\alpha\beta\gamma\delta} C^{\alpha\beta\gamma\delta} \right).$$
(4.56)

It is important to remember that the Einstein frame has no physical significance here; it is in the sense of the theory translated back into the brane frame that we think of those corrections. But working in terms of (4.56) we may think of those corrections in the same way as we have done so far and compare them with the case of standard slow-roll inflation on the brane.

Following the work of [16], in the low-energy limit, the differential equation for the scalar modes may be expressed as:

$$u'' + \left(\hat{c}_s^2 k^2 - \frac{\hat{\beta}}{\tau^2}\right) u = 0,$$
(4.57)

where u is related to the curvature by $u = \frac{1}{\dot{\varphi}_0} (1 + \alpha \hat{z}) \Psi$, with notation:

$$\epsilon = -\frac{\dot{H}}{H^2}, \ \lambda^{(n)} = \frac{d^n \ln \epsilon}{d \ln a^n}, \tag{4.58}$$

$$\hat{z} = \frac{4}{3} \frac{L^2 k^2}{a^2} + 2L^2 H^2 \left(1 + 3\epsilon - \lambda^{(1)} \right), \tag{4.59}$$

$$\hat{c}_s^2 = 1 + \frac{8}{3} \alpha L^2 H^2 \left(3 + 2\epsilon\right), \qquad (4.60)$$

$$\hat{\beta} = -\frac{a^2 H^2 \tau^2}{4} \left(2\lambda^{(2)} - \left(2 + \lambda^{(1)} \right) \left(2\epsilon + \lambda^{(1)} \right) \right), \qquad (4.61)$$

$$+ 2\alpha L^2 a^2 \tau^2 H^4 \left(\lambda^{(3)} - \lambda^{(2)} - \lambda^{(1)} \lambda^{(2)} - 6\lambda^{(2)} \epsilon, + 2\epsilon + 4\epsilon^2 - 6\epsilon^3 + 3\epsilon\lambda^{(1)} + 11\epsilon^2\lambda^{(1)} \right).$$

Using the same arguments as in section 3.3, eq.(4.57) will admit a nearly scale-invariant solution if the variation of $\hat{\beta}$ is negligible and if $\hat{\beta} \ll 1$. If the parameter ϵ is considered constant as a first approximation (we set all the $\lambda^{(n)}$ to zero to start with), then by integration, $H = \frac{1}{(\epsilon - 1)a\tau}$. The expression for $\hat{\beta}$ simplifies to:

$$\hat{\beta} = \frac{\epsilon}{(\epsilon - 1)^2} \left[1 + 4\alpha L^2 H^2 \left(1 + 2\epsilon - 3\epsilon^2 \right) \right]$$
(4.62)

$$H \sim \tau^{-\frac{\epsilon}{\epsilon-1}}.\tag{4.63}$$

Setting the constant α to zero in this expression gives rise to the standard result: $\hat{\beta}$ remains a small constant $\hat{\beta} \ll 1$ for both $\epsilon \ll 1$ and $\epsilon \gg 1$. However, adding the C^2 -corrections in the effective theory gives rise to the slightly more complicated situation: In the standard slow-roll inflation case as studied in the previous section $\epsilon \ll 1$, so that $\hat{\beta} \simeq \epsilon (1 + 4\alpha L^2 H^2)$, with $H \simeq$ constant during inflation, giving rise to the almost scale-invariant spectrum as discussed previously.

For $\epsilon \gg 1$, on the other hand,

$$\epsilon_{\rm F} = \frac{1}{2\epsilon} \ll 1, \tag{4.64}$$

$$\hat{\beta} = 2 \epsilon_{\rm F} \left(1 - 3\alpha L^2 H^2 \epsilon_{\rm F}^{-2} \right), \qquad (4.65)$$

where $\epsilon_{\rm F}$ denotes the fast-roll parameter as introduced in [16].

Two important problems arise here. First of all, $\alpha L^2 H^2$ should be of the same order of magnitude or smaller than $\epsilon_{\rm F}^2$ to avoid the correction term to become large. If not, the procedure breaks down as the correction becomes more important than the "leading" term and higher order terms in α have to be introduced as well as other corrections in the Weyl tensor. In the low-energy limit the term $L^2 H^2$ will indeed be small, but not necessarily small compared to $\epsilon_{\rm F}$. The second departure from the standard result arises from the τ dependence of the correction term. In the context of fast-roll, the Hubble parameter is not constant but varies as $H \sim \tau^{-1}$. In general the corrections to $\hat{\beta}$ will not be constant and it will not be possible to follow the usual derivation from eq.(3.52) to get the expression (3.55) and find a scale invariant power spectrum.

If in the Cyclic scenario, the large scale structure is produced while the Hubble parameter is still tiny and its variations small enough for the quantity $L^2 H^2 \epsilon_{\rm F}^{-2}$ to remain small and constant during the process. Those corrections will therefore have a negligible effect on the power spectrum of the perturbations. It will be consistent to keep treating $\hat{\beta}$ as a small constant, giving rise to the almost scale-invariant spectrum:

$$n_{S, \text{ fast-roll}} - 1 = -4\epsilon_{\rm F} - 4\eta_{\rm F} + 12 \left. \alpha L^2 H^2 \epsilon_{\rm F}^{-1} \right|_{\tau^*}, \qquad (4.66)$$

with the second fast-roll parameter as defined in [16]: $\eta_{\rm F} = 1 - \eta \epsilon_{\rm F}$.

However in a more general case of braneworld cosmology, if we consider a general potential satisfying fast-roll conditions, we will generically expect to see some departures from a nearly-scale invariant spectrum when some C^2 -corrections are introduced in the Weyl tensor.

4.4 Discussion

In this section we have presented a consistent way to modify the low-energy effective theory in order to accommodate corrections coming from the first Kaluza Klein mode. In order to perform this task, we have explained how exactly these corrections could come in the low-energy effective theory and have presented the different properties these corrections should have. We have then focused to a specific example for which the correction term is a function of the metric only and can be derived from an action. In that case a specific ansatz has been suggested and checked in the quasi-static limit. In that limit, the perturbations around static branes can be worked out from a five-dimensional point of view giving a solid consistency check for our ansatz. Comparing this exact result with the proposed ansatz allows us to fix the undetermined coefficients and to give a strong motivation for the study of this ansatz to other braneworld cosmological scenarios. In particular, we have concentrated our study on two particular examples.

First we analysed the contribution of this correction term (introduced with a coefficient α) in the model of brane inflation of the previous chapter, where again the typical energy scale was important compared to the brane tension. Several new features were observed which, as far as we are aware, extend previous results in chaotic brane inflation. First of all, both tensor and scalar modes propagate at a speed different from the speed of light. The corrections bring a new mode-dependent term in the amplitude of both tensor and scalar perturbations which does not compensate in their ratio and which could be important at high energies. There is as well a new mode-dependent term in the estimate of the nongaussianity which is proportional to the small coefficient α . This term is not damped by any slow-roll parameter. This is a critical feature, addressing the issue of the importance of non-gaussianity when those corrections are introduced. Our prescription only makes sense when the constant α is small, so that we can not address the issue of non-gaussianity outside this regime, however there could be some situations for which the new term in the non-gaussianity estimate might be the leading one. In that case, the cubic terms might not be small compared to the quadratic ones, and a perturbative approach might not be sensible. These are important features that have to be considered seriously in order to distinguish between purely four-dimensional inflation and brane inflation. Comparing these results with observations might give a constraint on the order of magnitude of the constant α and hence on the distance d between the branes.

In the second example, we used the formalism to study how the duality, relating density perturbations in expanding and contracting Friedmann cosmologies, was affected by the introduction of this correction. We compared the production of a scale-invariant spectrum in a model of slow-roll inflation where the typical energy scale was much smaller than the brane tension, with an Ekpyrotic or Cyclic model for which the scalar field was evolving in a potential satisfying fast-roll conditions. The first order corrections, proportional to the constant α , become negligible in the slow-roll limit but could be large in a general fast-roll limit. We therefore recover the nearly scale-invariant spectrum in the slow-roll inflation but the situation becomes more complicated in a general fast-roll scenario. In this case, the departure from a scale-invariant spectrum could become more important when these corrections interfere. This is an important new result enabling us to differentiate between the fast-roll and slow-roll scenarios. Slow-roll inflation can be considered both in a purely four-dimensional context and in a model of braneworld cosmology. Fast-roll models, on the other hand, require a contracting Universe and only make sense when the intrinsic theory is higher-dimensional. In that situation, KK corrections cannot be avoided. It is therefore vital to understand in detail the consequences of those modes for this model as we have here attempted to do.

The main limitation of this study is the derivation of the correction term which has been introduced as an ansatz and checked only in a specific limit. In the end of the next chapter we shall give another argument for the study of this correction term in the close-brane limit, which provides an extra motivation, but does not confirm the validity of our ansatz in the one-brane limit of the model. We have tried however to go beyond the usual assumptions and incorporate some effects of the brane nature of the theory. To be completely rigorous, one should ultimately try to attack the five-dimensional problem directly, which shall be done in the close-brane limit in next chapter. Nonetheless, we hope that our formalism can be used to give greater analytical insight into typical braneworld effects.

Chapter 5

Close-brane Effective Theory

5.1 Introduction



Nut, Sky Goddess (Egypt)

In chapter 2, the existence of an effective four-dimensional theory as been pointed out and derived at low energy. Beyond this limit, however, braneworld models differ remarkably from standard four-dimensional models and have some distinguishing features which could either generate cosmological signatures or provide alternative scenarios to standard cosmology. In particular, in the limit where two three-branes are close to each other, characteristic properties of the model may be determined and analysed, which will be the focus of this chapter. Near the brane collision, the low-energy effective theory predicts that the Hubble constant on each brane is related to the proper contraction or expansion velocity of the fifth dimension \dot{d} by $L^2 H_{\pm}^2 = \dot{d}^2/4$. However, an exact calculation gives the result $L^2 H_{\pm}^2 = \tanh^2 \dot{d}/2$. As expected, this agrees with the four-dimensional effective theory only at low velocities. To lowest order in velocity, the low-energy limit gives an accurate result for the brane geometries. The aim of this chapter is to go beyond the lowenergy limit and to develop a covariant formalism which describes these velocity corrections exactly in the small distance limit $d \ll L$.

In chapter 3, we mentioned that in some scenarios of brane inflation [129, 131, 139, 153], the potential could be important compared to the brane tension and some different constraints for the potential can hence be considered. In a scenario where the potential is steep, for instance, an alternative explanation for the end of inflation can be given [139]. In the Cyclic Universe [86,87], as another example, it is suggested that terms of fourth order in velocities should be considered in order to obtain a scale invariant spectrum after the bounce [154]. To work with such models, it seems important to understand the behaviour of these high-energy corrections and in particular to understand their consequences for the production of the large scale structure, as much in the context of an expanding than in a contracting Universe. Part of this chapter will therefore be devoted to the development of a covariant formalism capable of describing these velocity corrections.

However, in order find a solvable theory when the low-energy constraint is relaxed, we need to work in another limit. For the purpose of this chapter, we choose to work in the special limit where the distance between the branes is much smaller than the length scale L characteristic for the five-dimensional AdS bulk. We stress that this is not equivalent to taking a Minkowski limit. As we shall see, the resulting effective theory is still sensitive to the bulk curvature. This limit is interesting in cosmology for two main reasons. Firstly, there has been a considerable interest in the interpretation of the Big Bang as a brane collision [86, 87, 119, 154–156], or as a collision of bubbles [157]. In such a model, we therefore expect to find a regime for which the distance between the branes is very small just before or just after the Big Bang. Hence, such a limit seems relevant for the study of perturbations.

Furthermore, it is in the limit where the distance between the branes is small that we

expect the geometry on the branes to be well-described by a four-dimensional theory, as we have seen in chapter 2. This provides us with a second reason motivating the study of the model in the close-brane regime. Indeed, when the branes are far apart, the excitation of the bulk degree of freedom requires less energy. This allows many Kaluza Klein modes to affect the brane geometry (even a continuum of them for an important brane separation), making the theory on the brane non-local [149, 158].

Following these arguments, we shall hence focus on the close-brane limit. To work in this regime, we follow part of the idea of T. Shiromizu *et.al.* in [112] although our final result will be different. The main idea of this paper is to express the extrinsic curvature on the negative-tension brane as a Taylor expansion in terms of the extrinsic curvature on the positive-tension brane and its derivatives along the normal direction. We can then use the five-dimensional Einstein equations to express any second (or higher) order normal derivatives of the extrinsic curvature in terms of first normal derivatives and derivatives along the four transverse directions. Since the normal derivative of the extrinsic curvature on the positive-tension brane is known up to the induced Weyl tensor on that brane, this gives a formal equation for the induced Weyl tensor on the positive-tension brane, which is the only unknown information on the brane.

As we might expect, the exact expression for the Weyl tensor in the close-brane limit has some higher derivative correction terms which are not present in the low-energy fourdimensional effective theory. However we may check that, at low velocities, this expression for the Weyl tensor is consistent with the one derived from the low-energy effective theory in the small d limit. Furthermore, as a non-trivial check we may verify that our result gives precisely the correct result to leading order in d for the background solution where it is possible to solve the five-dimensional geometry exactly. However the prescription is completely covariant and will allow us to study perturbations without needing to solve the full five-dimensional equations.

Having checked the consistency of this effective theory for the background, we use it in order to study the effect of matter perturbations about an empty background, (*ie.* a 'stiff source' approximation). For this we consider the branes to be empty for the background and

introduce matter on the positive-tension brane only at the perturbed level. We then study in detail the propagation of tensor and scalar perturbations. Although the five-dimensional nature of the theory does not affect the way tensor and scalar perturbations propagate in a given background, the coupling to matter is indeed affected. In particular, we show that the effective four-dimensional Newtonian constant depends both on the distance between the branes and their rate of separation. We then extend the analysis in order to have a better insight of what might happen when the small brane separation condition is relaxed.

As mentioned in chapter 1, one of the most promising routes of braneworld cosmology is the possibility of interpreting the radion scalar field as playing the rôle of the inflaton scalar field on the positive-tension brane [66, 78, 159]. During inflation both branes could be moving apart. When the negative-tension brane moves towards infinity, its effect on the positive-tension brane becomes negligible which means that the scalar field decouples, giving an explanation of why such a field is not seen on the brane at the present time. In order to interpret the radion as a candidate for the inflaton a potential for the radion has to be introduced as explained in chapter 2. After reviewing how the introduction of a potential can be generalised to this close-brane effective theory, we can therefore study the propagation of perturbations in a background evolving with a potential for the radion. We then study to which extent the radion can be interpreted as a possible candidate for the inflation and show what would be the characteristic features (if any) present in the power spectrum.

Finally, we have shown in chapter 4, that in the quasi-static limit, the first Kaluza Klein mode could be correctly modelled by the presence of a term $A_{\mu\nu}$ in the Weyl tensor. We have shown that $A_{\mu\nu}$ could be obtained from an action and was satisfying all the required condition to model a Kaluza Klein correction. However, the introduction of such a term was only an ansatz which was checked to be consistent in the quasi-static limit. In the close-brane limit, the validity of this ansatz can be tested again. In particular, an exact covariant expression for this term can be computed and checked against $A_{\mu\nu}$.

But first we shall start this chapter by pointing out the main discrepancy between the effective low-energy theory and the exact five-dimensional result, giving a motivation to go

beyond the low-energy limit.

5.2 Test of the Low-energy Effective Theory in the Close-brane Limit

5.2.1 Five-dimensional Background Behaviour

Following the background study of section 2.4, we find the five-dimensional background solution for the two-brane RS model in the frame where the bulk was static (2.20). We now consider the specific case where the branes are moving apart after a collision and the aim of this section will be to relate the Hubble parameter on the branes to the velocity of the distance between the branes in the close-brane regime. We suppose that the collision occurs at $\tau = 0$ and denote by a_0 the value of the scale factor when the branes coincide (the situation where the branes are moving towards each other before the collision is completely analogous).

In contrast to the exact treatment of section 2.4, we now only consider the system in the limit of small brane separation, *ie.* $a_+ \approx a_- \approx a_0$, $n_+ \approx n_- \approx n_0$. In what follows we will describe only the leading term in d/L and denote by ' \approx ' the leading order in this expansion. Using (2.23), to linear order we have:

$$Y_{\pm}(T) \approx Y_0 \mp v_{\pm} (T - T_0)$$
 (5.1)

$$v_{\pm} = n_0 \sqrt{1 - \frac{n_0^2}{a_0^2} \left(1 \pm \frac{\kappa L}{6} \rho_{\pm}\right)^{-2}}, \qquad (5.2)$$

where the collision happens at $T = T_0$ and $Y = Y_0$. When the branes are close, the proper distance between the branes goes as $d \sim (T - T_0)$, so the corrections are of order $\mathcal{O}\left(\frac{T-T_0}{L}\right) = \mathcal{O}\left(\frac{d}{L}\right)$. This is the limit in which we will work all through this chapter.

So far we have worked in the frame where the bulk was static. However if we are interested in the proper distance between the branes, it is more intuitive to derive it from the frame where the branes are static. The gauge transformation between the frame (2.20)
where the bulk is static and a frame of the form (2.1) where the branes are static is in general complicated. However, for the purpose of this study, it is enough to consider the frame in which the branes are "quasi-static", or static to leading order in d/L or in $(T - T_0)/L$. In order to work in such a frame, we may perform the gauge transformation $(Y,T) \rightarrow (y,t)$:

$$T = T_0 + \frac{t}{n_0} \cosh \alpha(y) \tag{5.3}$$

$$Y = Y_0 + t \sinh \alpha(y) \tag{5.4}$$

$$\alpha(y) = y \tanh^{-1}\left(\frac{v_{-}}{n_{0}}\right) - (1-y) \tanh^{-1}\left(\frac{v_{+}}{n_{0}}\right).$$
(5.5)

We can indeed check that at y = 0, $Y = Y_0 - v_+(T - T_0)$ and at y = 1, $Y = Y_0 + v_-(T - T_0)$. In this new frame, the branes are static to leading order in d and located at y = 0 and y = 1. In terms of the new coordinates y and t, the bulk geometry in the limit of close separation is:

$$ds^{2} = A(t)^{2} dy^{2} - dt^{2} + a^{2} d\mathbf{x}^{2}, \qquad (5.6)$$

with
$$A(t) = t \left(\tanh^{-1} \left(\frac{v_+}{n_0} \right) + \tanh^{-1} \left(\frac{v_-}{n_0} \right) \right),$$
 (5.7)

where we have treated v_{\pm} as constants $v_{\pm} \approx n_0 \sqrt{1 - \frac{n_0^2}{a_0^2} \left(1 \pm \frac{\kappa L}{6} \rho_{\pm}(T_0)\right)^{-2}}$. In this limit, the induced metric on the branes is then $ds_{\pm}^2 = -dt^2 + a(t)^2 d\mathbf{x}^2$, where t is the proper time. When $d \ll L$, the proper distance between the branes is given by:

$$d \approx \int_{0}^{1} A(t) \, \mathrm{d}y$$
$$\approx t \left(\tanh^{-1} \left(\frac{v_{+}}{n_{0}} \right) + \tanh^{-1} \left(\frac{v_{-}}{n_{0}} \right) \right), \qquad (5.8)$$

so that, to leading order, the expansion of the fifth dimension with respect to the proper time t is

$$\dot{d} \approx \tanh^{-1}\left(\frac{v_+}{n_0}\right) + \tanh^{-1}\left(\frac{v_-}{n_0}\right),$$
(5.9)

where, here and in what follows a dot represents a derivative with respect to the proper time t. At the collision, v_{\pm} is related to the Hubble parameter (2.27) by:

$$\frac{v_{\pm}^2}{n_0^2} \approx 1 - \frac{n_0^2}{a_0^2} \left(1 \pm \frac{\kappa L}{6} \,\rho_{\pm}(T_0) \right)^{-2} \approx L^2 H_{\pm}^2(\tau = 0) \left(1 \pm \frac{\kappa L}{6} \,\rho_{\pm}(T_0) \right)^{-2}, \tag{5.10}$$

giving the relation between the Hubble parameter on the positive tension brane and d:

$$\dot{d} \approx \tanh^{-1}\left(\frac{LH_{+}}{1+\frac{\kappa L}{6}\rho_{+}}\right) + \tanh^{-1}\left(\frac{\sqrt{L^{2}H_{+}^{2} - \frac{\kappa L}{3}\left(\rho_{+} + \rho_{-}\right) - \frac{\kappa^{2}L^{2}}{36}\left(\rho_{+}^{2} - \rho_{-}^{2}\right)}}{1 - \frac{\kappa L}{6}\rho_{-}}\right), (5.11)$$

which can be inverted as

$$LH_+ \approx \tanh\left(\frac{\dot{d}}{2}\right) + \frac{\kappa L}{6}\rho_+^{\text{eff}}$$
 (5.12)

$$\rho_{+}^{\text{eff}} = \frac{1}{\sinh \dot{d}} \left(\cosh \dot{d} \,\rho_{+} + \rho_{-} \right). \tag{5.13}$$

This result is derived directly from the full five-dimensional equations subject only to the assumption that the branes are close, so this result is only valid just before or just after a collision. In the absence of matter, we can point out that the Hubble parameter is bounded $L^2H_{\pm}^2 < 1$ which is consistent with (2.27) when $\rho_{\pm} = 0$. The expression of the Hubble parameter on the negative tension brane and the expansion of the fifth dimension can be derived in a similar way. We obtain the analogous result

$$LH_{-} \approx \tanh\left(\frac{\dot{d}}{2}\right) - \frac{\kappa L}{6}\rho_{-}^{\text{eff}},$$
(5.14)

with $\rho_{-}^{\text{eff}} = \left(\cosh \dot{d} \rho_{-} + \rho_{+}\right) / \sinh \dot{d}.$

We can compare this result with the analogous relation derived from the standard lowenergy effective theory derived in 2.6. In Section 5.4.1 we will then compare this result with the one obtained from the close-brane theory, derived in this chapter and we shall see that they agree perfectly.

5.2.2 Comparison with the Low-energy Theory

We may now examine the behaviour of the effective theory in the special case of a flat FRW background as in 2.6.4. We consider the special case where the positive-tension brane expands. As a consequence of the tracelessness of the Weyl tensor, we have already mentioned in (2.50) that the Friedmann equation on the brane was of the same form as (2.28), with an integration constant \tilde{D} which has to be expressed in terms of the distance between the branes. Using the 00-component of the modified Einstein equation in the low-energy effective theory (2.45), we find the expression for the constant \tilde{D}_+ in terms of \dot{d} :

$$\frac{\tilde{D}_{+}}{L^{2}a_{+}^{4}} = \frac{\Psi^{2}}{1 - \Psi^{2}} \left(-2\frac{\dot{d}}{L}H_{+} + \frac{\dot{d}^{2}}{L^{2}} + \frac{\kappa}{3L}\left(\rho_{+} + \rho_{-}\right) \right).$$
(5.15)

Substituting this expression in the modified Friedmann equation (2.50), we therefore have

$$(1-\Psi^2)H_+^2 + 2\Psi^2\frac{\dot{d}}{L}H_+ - \Psi^2\frac{\dot{d}^2}{L^2} = \frac{\kappa}{3L}\left(\rho_+ + \Psi^2\rho_-\right).$$
(5.16)

This is a quadratic equation with the two solutions:

$$LH_{+} = \frac{\dot{d}}{(1-\Psi^{2})} \left(-\Psi^{2} \pm \sqrt{\Psi^{2} + \frac{\kappa L}{3\dot{d}^{2}} \left(1-\Psi^{2}\right) \left(\rho_{+} + \Psi^{2}\rho_{-}\right)} \right).$$
(5.17)

The two possible signs correspond to the branes moving either in the same (-) or opposite (+) directions, which can be seen as follows. For empty branes, the Friedmann equation simplifies to

$$LH_{+} = \frac{\dot{d}}{1 \pm \Psi^{-1}}.$$
(5.18)

So for empty branes, using the conformal relation (2.37) (*ie.* $a_{-} = \Psi a_{+}$), the Hubble parameter on the negative-tension brane can be written in terms of H_{+} as:

$$H_{-} \equiv \frac{\dot{a}_{-}}{a_{-}} = -\frac{\dot{d}}{L} + H_{+} = \mp e^{d/L} H_{+}$$

= $\mp H_{+} \left(1 + \mathcal{O}(d/L) \right).$ (5.19)

So the - sign in (5.18) or in (5.17) corresponds to the situation where $H_{-} \sim H_{+}$ for which the branes are moving in the same direction. In that case, near the collision,

$$\dot{d} \approx -d H_+,\tag{5.20}$$

so the branes take an infinite time to collide. Instead, we are interested in the situation where the collision happens at finite time and the branes are moving in opposite direction $H_{-} \simeq -H_{+}$, corresponding to a + sign in (5.18) or in (5.17). This is the situation we consider for branes moving apart just after a collision. In this case, taking the close-brane limit of the Friedmann equation (5.17), we get

$$H_{+} \approx \frac{\dot{d}}{2L} + \frac{\kappa}{6\dot{d}} \left(\rho_{+} + \rho_{-}\right), \quad \text{for } d \ll L.$$
(5.21)

We notice that both H_+ and d are finite at the collision,

$$\dot{d} \sim LH_+ \sim \left(d/L\right)^0. \tag{5.22}$$

We may compare this result with (5.12), which is correct to all orders in \dot{d} for small d/L. As expected, the low-energy theory reproduces only the leading term in \dot{d} and gravity seems to couple the same way to the matter on both branes. This is due to the fact that the effective low-energy theory neglects part of the contribution from the Weyl tensor which depends on the matter content of both branes. The effective theory is therefore only valid at low-energies ($\rho \ll 1/\kappa L$) as already mentioned in section 2.6.4 and only to leading order in velocities (even for the background). We may point out that in this result both LH_{\pm} and \dot{d} appear to be unbounded. This is only true in this low-energy regime for which the restrictions are very strong $LH_{\pm} \sim \dot{d} \ll 1$. As seen in (5.12) when $LH_{\pm} \sim 1$ some new restrictions have to be imposed.

Since the low-energy effective theory only predicts the leading order of velocity, it might be possible to go beyond the low-energy restriction and find a theory which would be valid to all order in velocities. In order to derive such a theory, we will work in a regime where the branes are close to each other. In the rest of this chapter, we show the existence and consistency of such a theory which successfully reproduces (5.12) and agrees with the low-energy theory in the regime of small separation *and* velocity in which they are both valid.

5.3 Derivation of a Close-brane Covariant Theory

5.3.1 Toy model

In order to understand the procedure we will follow to work out the theory on the brane in the close-brane limit, we first examine the following one-dimensional example. We consider a second order differential equation:

$$f''(y) = U(y)f'(y) + V(y)f(y) + W(y),$$
(5.23)

where U, V and W are some known function of y.

We assume that the function f is known at y = 0 and at y = 1, and we wish to find the value f'(0) and f'(1) of its derivative on the boundaries. One way to do it would be to solve the differential equation exactly with the two boundary conditions for f. Once f(y) is known for any y, we can infer f'(0) and f'(1). But this method would be equivalent to solving the five-dimensional Einstein equations exactly in order to obtain the induced geometry on the brane. Although this is in theory possible it would be very hard to do. Instead we will summarise in this example the method used by [112]. The idea is not to solve the differential equation exactly but to differentiate it in order to use it in the Taylor expansion:

$$f(y=1) = \sum_{n \ge 0} \frac{1}{n!} f^{(n)}(0).$$
(5.24)

By differentiating equation (5.23), we can find an expression for $f^{(n)}(y)$:

$$f^{(n+2)}(y) = U_n(y)f'(y) + V_n(y)f(y) + W_n(y),$$
(5.25)

where U_n, V_n and W_n may be found in a recursive way:

$$U_{n+1}(y) = U'_n(y) + V_n(y) + U(y)U_n(y), \quad \text{with} \quad U_0 = U(y), U_{-1} = 1, U_{-2} = 0$$
$$V_{n+1}(y) = V'_n(y) + V(y)U_n(y), \quad \text{with} \quad V_0 = V(y), V_{-1} = 0, V_{-2} = 1$$
$$W_{n+1}(y) = W'_n(y) + W(y)U_n(y), \quad \text{with} \quad W_0 = W(y), W_{-1} = 0, W_{-2} = 0$$

Using these expressions, we may write f'(0) as:

$$f'(0) = \frac{1}{\sum_{n\geq 0} \frac{1}{n!} U_{n-2}(0)} \left(f(1) - \sum_{n\geq 0} \frac{1}{n!} \left(V_{n-2}(0) f(0) - W_{n-2}(0) \right) \right).$$
(5.26)

Knowing U_n, V_n and W_n in a recursive way, we can perform the sums and find an exact expression for f'(0). We may point out that this method actually allows us to find f'(y)for any y between the boundaries. This will be very similar to the method we will use to find the induced Weyl tensor on the brane. Although the extrinsic curvature is known on the branes, its normal derivative (which involves the Weyl tensor) is not. We can derive, however, a second order differential equation for the extrinsic curvature which allows us to calculate this derivative in the same way as (5.26).

However, already in this linear one dimensional problem, the recursive relations are non-trivial. The five-dimensional problem is even harder since, unsurprisingly, the equations are non linear and formidably complicated. However, if we keep only the leading terms in d/L, we find that the second order differential equation for $K_{\mu\nu}$ is linear and, remarkably, the Taylor series corresponding to (5.26) becomes tractable. We therefore end up with an expression, correct to leading order in d/L, for the normal derivative of the extrinsic curvature on the positive-tension brane which enables us to write down the Einstein equations and obtain the close-brane limit of the exact theory on the brane.

5.3.2 Regime of validity

From now on we will work in a regime where the branes are very close $d \ll L$. As already mentioned in section (5.2.2), there are two types of solution for the background, depending on whether the branes are moving in the same or in opposite directions. If the branes are moving in the same direction, we have seen in (5.20) that, for the background, $\dot{d} \sim d/L$. It is subject to this assumption that [112] was derived. In this regime, we may check that to leading order in d, they recover the low-energy effective theory. Therefore, for this regime, the low-energy effective theory is valid to all order in velocities (at small d). However from (5.22) we see that this is not valid when the branes are moving in opposite directions. In that case, $\dot{d} \sim (d/L)^0 \sim 1$. This will be the solution we will be interested in for the rest of this work and we will assume the relation $\partial_{\mu}d \sim (d/L)^0$. Although this is strictly true only for the background, if we work with perturbations in a comoving gauge, this relation will still hold. In that gauge, $\partial_{\mu}d \sim (d/L)^0$ will still be true covariantly.

Furthermore, for adiabatic perturbations, the perturbations behave the same way as the most general background solution. For the background there are two kinds of solutions, one where the branes move in the same direction for which $\partial_{\mu}d \sim d/L$ and one where the collision happens at finite time, for which $\partial_{\mu}d \sim (d/L)^0$. This is true for adiabatic perturbations as well. The adiabatic perturbations will follow a similar evolution to one of the two background solutions or a superposition of them. Since the low-energy effective theory reproduces correctly one type of solution we may focus on perturbations that follow the other kind of behaviour for which $\partial_{\mu}d \sim (d/L)^0$.

At the level of perturbations, one might think that this procedure might break down since the perturbations diverge. But this divergence is actually logarithmic in d and is therefore negligible compared to the terms in 1/d that we will find in the theory (5.100). Compared to L/d, $\log(d/L) \sim (d/L)^0$. In [156], it is actually shown that in the right gauge, the perturbations remain "small" going towards the bounce.

5.3.3 Gauss-Codacci equations

In this subsection we will follow the formalism of [112]. In order to simplify the implementation of the Israël junction conditions, which would be otherwise difficult, we work in a frame (2.1) where the branes are assumed to be static, located at the fixed positions y = 0, 1:

$$ds^{2} = A^{2}(y, x)dy^{2} + q_{\mu\nu}(y, x)dx^{\mu}dx^{\nu}.$$
(5.27)

In what follows, we use the same notation as in section 2.3, in particular we recall that $q_{\mu\nu}^{(+)}(x) = q_{\mu\nu}(y=0,x)$ and $q_{\mu\nu}^{(-)}(x) = q_{\mu\nu}(y=1,x)$ are the induced metrics on both branes.

From the Gauss equations, the Einstein tensor on a y = const hypersurface is given by

(2.12)

$$G^{\mu}_{\nu}(y) = \frac{3}{L^2} \delta^{\mu}_{\nu} + K K^{\mu}_{\nu} - K^{\mu}_{\alpha} K^{\alpha}_{\nu} - \frac{1}{2} \delta^{\mu}_{\nu} \left(K^2 - K^{\beta}_{\alpha} K^{\alpha}_{\beta} \right) - E^{\mu}_{\nu}.$$
(5.28)

The unknown quantity in (5.28) is the electric part of the projected Weyl tensor E^{μ}_{ν} which is traceless:

$$R = -\frac{12}{L^2} - K^2 + K^{\beta}_{\alpha} K^{\alpha}_{\beta}.$$
 (5.29)

The Weyl tensor E^{μ}_{ν} can be expressed in terms of more recognisable quantities as (2.5)

$$E^{\mu}_{\nu} = -\frac{1}{A}\frac{\partial}{\partial y}K^{\mu}_{\nu} - \frac{1}{A}D^{\mu}D_{\nu}A - K^{\mu}_{\alpha}K^{\alpha}_{\nu} + \frac{1}{L^{2}}\delta^{\mu}_{\nu}.$$
 (5.30)

Gauge Choice

To start with we consider A, the yy-component of the metric in (5.27), to be independent of y. Such a gauge choice can indeed be picked for the background geometry of the closebrane limit (5.6), and we will see in section 5.3.6 that the final answer is not sensitive to the y-dependence of A. In that case the proper distance between the branes is: d(x) = A(x). We can now rewrite the expression for the modified Einstein equation (5.28) in the form

$$G^{\mu}_{\nu} = \frac{1}{d} \left[D^{\mu} D_{\nu} d + \frac{\partial}{\partial y} K^{\mu}_{\nu} + \frac{2d}{L^2} \delta^{\mu}_{\nu} + dK K^{\mu}_{\nu} - \frac{d}{2} \delta^{\mu}_{\nu} \left(K^2 - K^{\beta}_{\alpha} K^{\alpha}_{\beta} \right) \right], \tag{5.31}$$

where the three last terms are negligible in the small distance limit $d \ll L$, and where $\frac{\partial}{\partial y}K^{\mu}_{\nu}$ is the only quantity undetermined on the brane (the extrinsic curvature itself is indeed determined through the Israël matching condition (2.17)). In order to find an expression for the derivative of the extrinsic curvature we will use a similar procedure to the toy model.

5.3.4 Taylor expansion of the extrinsic curvature on the branes

Knowing the extrinsic curvature on both branes, we write an expression for its derivative using the Taylor expansion:

$$K^{\mu}_{\nu}(y=1) = \sum_{n\geq 0} \frac{1}{n!} \left. \partial^{(n)}_{y} K^{\mu}_{\nu} \right|_{y=0}, \qquad (5.32)$$

and we will use the notation: $Q' \equiv \partial_y Q$ and $Q^{(n)} \equiv \partial_y^{(n)} Q$ for any quantity Q(y, x) carrying indices only along the directions of a y = const hypersurface.

In order to find the expression for $K^{\mu(n)}_{\nu}$ in (5.32) we need the derivative of the Weyl tensor and of the Christoffel Symbol. The last one is given by (2.30)

$$\Gamma^{\alpha \prime}_{\mu\nu} = D_{\mu}(d \, K^{\alpha}_{\nu}) + D_{\nu}(d \, K^{\alpha}_{\mu}) - D^{\alpha}(d \, K_{\mu\nu}).$$
(5.33)

For the Weyl tensor derivative, we may use the result derived in (2.33)

$$E^{\mu\nu}_{\nu} = d \Big(2K^{\alpha}_{\nu} E^{\mu}_{\alpha} - \frac{3}{2} K E^{\mu}_{\nu} - \frac{1}{2} K^{\alpha}_{\beta} E^{\beta}_{\alpha} \delta^{\mu}_{\nu} + C^{\mu}_{\ \alpha\nu\beta} K^{\alpha\beta} + (K^{3})^{\mu}_{\nu} \Big)$$

$$- \frac{1}{2d} D^{\alpha} \Big[d^{2} D^{\mu} K_{\alpha\nu} + d^{2} D_{\nu} K^{\mu}_{\alpha} - 2d^{2} D_{\alpha} K^{\mu}_{\nu} \Big] .$$
(5.34)

Both these relations (5.33) and (5.34) are valid for any y in the gauge where A is yindependent. The important point is that the system is now closed. The second derivative of the extrinsic curvature can indeed be expressed in terms of $E_{\mu\nu}$ and $E'_{\mu\nu}$ which themselves can be expressed back in terms of the extrinsic curvature and its first y-derivative. We obtain a second order differential equation for the extrinsic curvature which is non-linear but involves four-dimensional quantities only. We may therefore apply the procedure of section 5.3.1. So far all the results are exact and valid for any distance between the branes. However in order to be able to formally sum the Taylor expansion (5.32), we will, from now, on assume that the branes are close to one another and keep only the leading order in d in the expansion. The purpose of the next subsection will be to show a recursive relation for the derivatives of the extrinsic curvature on the brane $K^{\mu(n)}_{\nu}(0)$ in the small d limit. The sum of all these terms in the Taylor expansion will then be straight-forward.

5.3.5 Recursive relation for the derivative of the extrinsic curvature in the close-brane regime

In what follows we denote by $\mathcal{K}^{\mu(n)}_{\nu}(x)$ the leading order in d/L of $K^{\mu(n)}_{\nu}(y=0,x)$, symbolically,

$$K^{(n)}(y=0,x) = \mathcal{K}^{(n)}(x) \left(1 + \mathcal{O}(d/L)\right).$$
(5.35)

Using the small distance approximation, we rewrite the expansion (5.32) as:

$$K^{\mu}_{\nu}(y=1,x) \approx \sum_{n\geq 0} \frac{1}{n!} \mathcal{K}^{\mu(n)}_{\nu}(x),$$
 (5.36)

where we recall that \approx designates the leading order in the expansion. In order to find a general expression for $\mathcal{K}^{(n)}$, we will work in a recursive way. We will show in the next subsections that we may express the n^{th} y-derivative of the extrinsic curvature in the recursive way: $\mathcal{K}^{\mu(n)}_{\nu} = \hat{O}\mathcal{K}^{\mu(n-2)}_{\nu}$, where \hat{O} is a four-dimensional operator that we shall calculate. In order to simplify the calculation, we will first start by considering empty branes, leaving the effect of matter for the final stage. If no matter is present on the branes, the Israël matching conditions on each brane (2.17) impose the extrinsic curvature to be:

$$K^{\mu}_{\nu}(y=1,x) = K^{\mu}_{\nu}(y=0,x) = -\frac{1}{L}\,\delta^{\mu}_{\nu},\tag{5.37}$$

such that when the branes are empty, the Taylor expansion (5.36) can actually be simplified to

$$\sum_{n\geq 1} \frac{1}{n!} \mathcal{K}_{\nu}^{\mu(n)}(x) = 0.$$
(5.38)

In order to find this recursive relation, we first start by expressing $\mathcal{K}^{\mu(n)}_{\nu}$ and the leading order of the Christoffel symbol in the n = 1 and n = 2 case.

Example for the n=1 and n=2 case

We first concentrate on the n = 1 and n = 2 cases in detail in order to gain insight; the technicalities of the general n case are left for later.

• For n = 1, the first derivative of the extrinsic curvature on the brane, can be found using (5.30) and (5.37):

$$K_{\nu}^{\mu\prime}(y=0) = -dE_{\nu}^{\mu} - D^{\mu}D_{\nu}d|_{y=0}.$$
(5.39)

Since $\partial_{\alpha}\partial_{\beta}d \sim \partial_{\alpha}d \sim d^0$ the second term goes as d^0 . In the effective theory, from (2.46), we can check that on the brane $E^{\mu}_{\nu} \sim (1-\Psi^2)^{-1} \sim (d/L)^{-1}$. Although

we have argued that, at high energy, the effective theory does not give the exact expression for the Weyl tensor, we have seen (at least for the background) that the behaviour is the same, differing only in corrections of higher order in the velocity. In particular E^{μ}_{ν} should go as d^{-1} at high energies as well (we will check later that this is indeed the case). We therefore have $K^{\mu'}_{\nu}(y=0) \sim d^0$. Furthermore, from (5.33), we have $\Gamma' \sim \partial d \sim d^0$ at y = 0, so the derivative of the extrinsic curvature has the same order of magnitude.

• For the second derivative we have:

$$K_{\nu}^{\mu \prime \prime}(y) = -d E_{\nu}^{\mu \prime} - q^{\mu \beta \prime} D_{\beta} D_{\nu} d + q^{\mu \beta} \Gamma_{\beta \nu}^{\alpha \prime} \partial_{\alpha} d - d \partial_{y} \left(K_{\alpha}^{\mu} K_{\nu}^{\alpha} \right).$$

On the positive-tension brane we may compare these terms with the ones from $K^{\mu\prime}_{\nu}(y=0)$ as they both contribute to the Taylor expansion (5.32)

terms from
$$K''$$
 terms from K'
 $dE^{\mu}_{\nu}{}' = \frac{4d^2}{L}E^{\mu}_{\nu} \ll dE^{\mu}_{\nu},$
 $q^{\mu\beta}{}'D_{\beta}D_{\nu}d = \frac{2d}{L}D^{\mu}D_{\nu}d \ll D^{\mu}D_{\nu}d,$
 $d\partial_y (K^{\mu}_{\alpha}K^{\alpha}_{\nu}) = -\frac{2d}{L}K^{\mu}_{\nu}{}' \ll K^{\mu}_{\nu}{}',$
(5.40)

where the relation $q^{\mu\beta\prime} = d\mathcal{L}_n q^{\mu\beta} = -2dK^{\mu\beta}$ has been used to simplify the expressions. When $d \ll L$, the only term which is not negligible in comparison to the terms present in $K'^{\mu}_{\ \nu}(y=0)$ is the one coming from the derivative of the Christoffel symbol:

$$q^{\mu\beta}\Gamma^{\alpha\prime}_{\beta\nu}(0)\partial_{\alpha}d = -\frac{1}{L}\left(2\partial^{\mu}d\partial_{\nu}d - (\partial d)^{2}\delta^{\mu}_{\nu}\right)$$

$$\sim K^{\mu\prime}_{\nu}(0).$$
(5.41)

This last term will give a contribution of the same order as $K^{\mu\prime}_{\nu}(y=0)$. We may as

well calculate the contribution from the second derivative of the Christoffel symbol:

$$\Gamma^{\alpha \ \prime\prime}_{\mu\nu}(y) = (d K^{\alpha \prime}_{\nu})_{;\mu} + (d K^{\alpha \prime}_{\mu})_{;\nu} - (d K^{\prime}_{\mu\nu})^{;\alpha}$$

$$+ d \left(\Gamma^{\alpha \prime}_{\mu\rho}K^{\rho}_{\nu} + \Gamma^{\alpha \prime}_{\nu\rho}K^{\rho}_{\mu} - 2\Gamma^{\rho \prime}_{\mu\nu}K^{\alpha}_{\rho}\right)$$

$$- d q^{\alpha\beta}q_{\mu\sigma} \left(\Gamma^{\sigma \prime}_{\beta\rho}K^{\rho}_{\nu} - \Gamma^{\rho \prime}_{\beta\nu}K^{\sigma}_{\rho}\right)$$

$$- d \left(q^{\alpha\beta}q_{\mu\sigma}\right)' \left(\Gamma^{\sigma}_{\beta\rho}K^{\rho}_{\nu} - \Gamma^{\rho}_{\beta\nu}K^{\sigma}_{\rho}\right).$$
(5.42)

On the brane the last line is of order $(d/L)^2$ and the two middle ones are of order d/L. The three last lines are therefore negligible in comparison to the first line which gives a contribution of order $\Gamma''(0) \sim \partial dK'(0) \sim d^0$:

$$\Gamma^{\alpha \prime \prime}_{\mu\nu}(0) \approx \partial_{\mu} d \, K^{\alpha \prime}_{\nu}(0) + \partial_{\nu} d \, K^{\alpha \prime}_{\mu}(0) - \partial^{\alpha} d \, K^{\prime}_{\mu\nu}(0), \qquad (5.43)$$

where terms of order d/L have been omitted.

• In the previous specific case, we have therefore shown that the leading terms in the derivative of the extrinsic curvature were of same order of magnitude for n = 1 and n = 2: $\mathcal{K}_{\nu}^{\mu(1)} = -dE_{\nu}^{\mu} - D^{\mu}D_{\nu}d \sim d^{0}$ and $\mathcal{K}_{\nu}^{\mu(2)} = -\frac{1}{L} \left(2\partial^{\mu}d\partial_{\nu}d - (\partial d)^{2}\delta_{\nu}^{\mu}\right) \sim d^{0}$. We therefore have $K^{(m)} \sim d^{0}$ for m = 0, 1, 2 and the same is valid for the derivative of the Christoffel symbols.

General n case

In what follows we will use a symbolic notation omitting any indices or coefficients. We write symbolically, $q_{\alpha\beta} = q$, $K^{\alpha}_{\beta} = K$, $\partial_{\alpha} = \partial$ and $\partial_y = \prime$. In particular, we have: q' = dqK. Using this notation and expressing E^{μ}_{ν} in terms of K^{μ}_{ν} in (5.34), we may symbolically write an equation for K^{μ}_{ν} :

$$K'' = d \left(\partial^2 + \Gamma \partial + \partial \Gamma + \Gamma^2\right) q' + dK^3 + dK$$

$$+ dKK' + q \partial d \left(\Gamma' + d\Gamma K + d\partial K\right) + \partial^2 dq',$$
(5.44)

where again by ∂ we designate a normal derivative (as opposed to covariant derivative) with respect to any coordinate x^{μ} along the four-dimensional y = const hypersurface. This expression is true for any y. The leading order in d/L in this expression comes from the term $q \partial d \Gamma' \sim d^0$. All the other terms are of order d.

• For n = 2, we have shown that on the brane

$$K^{(m)}(0) \sim \Gamma^{(m)}(0) \sim d^0 \quad \forall \ m \le n.$$
 (5.45)

and

$$\Gamma^{\alpha(m)}_{\mu\nu}(0) \approx \partial_{\mu}d \,\mathcal{K}^{\alpha(m-1)}_{\nu} + \partial_{\nu}d \,\mathcal{K}^{\alpha(m-1)}_{\mu} - \partial^{\alpha}d \,\mathcal{K}^{(m-1)}_{\mu\nu}, \quad \forall \ m \le n$$
(5.46)

$$\mathcal{K}^{\mu(n)}_{\nu} = q^{\beta\nu}\Gamma^{\alpha(n-1)}_{\beta\nu} \partial_{\alpha}d, \qquad (5.47)$$

where we used the notation $\mathcal{K}^{\mu(0)}_{\nu} = K^{\mu}_{\nu}(y=0) = -\frac{1}{L} \delta^{\mu}_{\nu}$. The last line translates the fact that the leading contribution of the derivative of the extrinsic curvature comes from the derivative of the Christoffel symbol uniquely.

• Now let us assume that these three relations (5.45 - 5.47) hold for a given $n \ge 2$. In particular, $\Gamma^{(n)}(0) \sim d^0$ and

$$q^{(m+1)}(0) \sim dK^{(m)} \quad \forall \ 0 \le m \le n,$$
 (5.48)

$$q(0) \sim d^0. \tag{5.49}$$

For the next order in n + 1 we have, using the symbolic relation (5.44),

$$\begin{aligned} K^{(n+1)} &= \partial_y^{(n-1)} K'' \\ &= \partial_y^{(n-1)} \Big[d \left(\partial^2 + \Gamma \partial + \partial \Gamma + \Gamma^2 \right) q' + dK^3 + dK + dKK' + \partial^2 d q' \\ &+ q \,\partial d \left(\Gamma' + d \,\Gamma K + d \,\partial K \right) \Big], \end{aligned} \tag{5.50}$$

with

$$\partial_y^{(n-1)} d \partial^2 q' \Big|_{y=0} = d\partial^2 \left[q^{(n)}(0) \right] \sim d.$$
 (5.51)

Similarly we may check that

$$\partial_{y}^{(n-1)} \left[d \left(\Gamma \partial + \partial \Gamma + \Gamma^{2} \right) q' + d K^{3} + d K \right] \Big|_{y=0} \sim d$$
(5.52)

$$\partial_{y}^{(n-1)} \left[d K K' + q \partial d \left(d \Gamma K + d \partial K \right) + \partial^{2} d q' \right] \Big|_{y=0} \sim d, \qquad (5.53)$$

and

$$\partial_{y}^{(n-1)} \left[q \Gamma' \partial d \right] \Big|_{y=0} = \partial d \sum_{m=0}^{n-1} C_{n-1}^{m} q^{(n-1-m)}(0) \Gamma^{(m+1)}(0), \qquad (5.54)$$

with $q^{(n-1-m)}(0)\Gamma^{(m+1)}(0) \sim d^1$ if m < n-1 and $q\Gamma^{(n)}(0) \sim d^0$ so that:

$$\partial_y^{(n-1)} \left[q \Gamma' \partial d \right] \Big|_{y=0} \sim d^0.$$
(5.55)

We have therefore shown that if (5.45) is true for a given n, then $K^{(n+1)}(0) \sim d^0$ and its leading contribution comes from the derivative of the Christoffel symbol uniquely so (5.47) is also valid at order n + 1.

• We now want to show that the recursive relations (5.45) and (5.46) are true for the Christoffel symbol as well. We have $K^{(m)}(0) \sim d^0$ for any $m \leq n+1$ and $\Gamma^{(m)}(0) \sim d^0$ for any $m \leq n$. Using the relation (5.33) for $\Gamma^{\alpha\prime}_{\mu\nu}$, we have:

$$\Gamma^{\alpha(n+1)}_{\mu\nu}(y) = (d K^{\alpha(n)}_{\nu})_{,\mu} + (d K^{\alpha(n)}_{\mu})_{,\nu} - q_{\mu\rho}(d K^{\rho(n)}_{\nu})^{,\alpha}$$

$$+ d \partial^{(n)}_{y} \left[\Gamma^{\alpha}_{\mu\rho} K^{\rho}_{\nu} + \Gamma^{\alpha}_{\nu\rho} K^{\rho}_{\mu} - 2\Gamma^{\rho}_{\mu\nu} K^{\alpha}_{\rho} - q^{\alpha\beta} q_{\mu\sigma} \left(\Gamma^{\sigma}_{\beta\rho} K^{\rho}_{\nu} - \Gamma^{\rho}_{\beta\nu} K^{\sigma}_{\rho} \right) \right]$$

$$- \sum_{m=0}^{n-1} C^{m}_{n-1} \left(d K^{\rho(m)}_{\nu} \right)_{,\beta} \partial^{(n-m)}_{y} \left(q^{\alpha\beta} q_{\mu\rho} \right),$$
(5.56)

where C_{n-1}^m is the Newtonian polynomial, and where $d \partial_y^{(n)}(\Gamma K) \sim d$ and $\partial_y^{(n-m)}(q^{\alpha\beta}q_{\mu\rho}) \sim d$ for n > m, so the two last sums go as d. Finally the first term goes as $(d K_{\nu}^{\alpha(n)})_{,\mu} = d_{,\mu}K_{\nu}^{\alpha(n)} + \mathcal{O}(d)$. The Christoffel symbol goes therefore as $\Gamma^{(n+1)} \sim d^0$ if (5.45) holds for a given n, and its leading contribution comes from

$$\Gamma^{\alpha\,(n+1)}_{\mu\nu}(0) \approx d_{,\mu} \mathcal{K}^{\alpha\,(n)}_{\nu} + d_{,\nu} \mathcal{K}^{\alpha\,(n)}_{\mu} - d^{,\alpha} \mathcal{K}^{(n)}_{\mu\nu}, \qquad (5.57)$$

the relation (5.46) therefore holds for n + 1.

• Assuming (5.45) and (5.46) to hold for a given n, we have therefore shown that these relations hold as well for n + 1. We can therefore finally conclude that

$$K^{(n)} \sim \Gamma^{(n)} \sim d^0 \quad \text{for any } n \ge 0, \tag{5.58}$$

and similarly, (5.46) is valid for any $n \ge 0$. Furthermore, we have seen through this development that the leading contribution for the extrinsic curvature comes from the derivative of the Christoffel symbol only and we have shown that the relation (5.47) is valid for any $n \ge 2$.

• To summarise, we have shown recursively that the leading term in the extrinsic curvature uniquely comes from the term

$$\mathcal{K}^{\mu(n)}_{\nu} = q^{\beta\nu} \Gamma^{\alpha(n-1)}_{\beta\nu} \partial_{\alpha} d
= \left(d^{\mu} \mathcal{K}^{\alpha(n-2)}_{\nu} + d_{,\nu} \mathcal{K}^{\alpha\mu(n-2)} - d^{,\alpha} \mathcal{K}^{\mu(n-2)}_{\nu} \right) d_{,\alpha}.$$
(5.59)

We therefore have a recursive expression for the leading order in d/L of the normal derivatives of the extrinsic curvature that we can use in order to formally sum the Taylor expansion (5.32) or (5.36). In particular we have shown the remarkably straight-forward recursive relation between the leading orders of the extrinsic curvature derivatives:

$$\mathcal{K}^{\mu\,(n)}_{\nu} = \hat{O}\mathcal{K}^{\mu\,(n-2)}_{\nu},\tag{5.60}$$

with the four-dimensional operator \hat{O} defined as

$$\hat{O}Z^{\mu}_{\nu} = \left[d^{,\mu}Z^{\alpha}_{\ \nu} + d_{,\nu}Z^{\alpha\mu} - d^{,\alpha}Z^{\mu}_{\ \nu}\right]d_{,\alpha},\tag{5.61}$$

for any four-dimensional tensor Z^{μ}_{ν} . This is a remarkable result since it will make the Taylor expansion (5.36) very easy to sum. We may note that in this summation, the term $\mathcal{K}^{\mu(n)}_{\nu}$ comes in with the same contribution as $\mathcal{K}^{\mu(n-2)}_{\nu}$, so all derivatives should bring an equivalent contribution to the Taylor expansion. Furthermore we can see that the operator \hat{O} is second order in derivative. So in a low-energy theory, the successive action of \hat{O} should be negligible, although this is not the case for the situation we consider.

For simplicity we exposed in detail the case for which no matter was present on the branes and $A^2 = g_{yy}$ was assumed to be *y*-independent. In the next section we shall see what happens when theses assumptions are relaxed. It is unclear whether a gauge for which A is *y*-independent can always be fixed, for a discussion of this gauge issue see [160].

Although our result seems to be completely gauge invariant, it has been derived in a gauge whose existence can not be completely justified. To be entirely consistent, we shall therefore show in the next subsection that assuming A to be y-independent does not affect the method we used nor the result in the close-brane regime. The extension to the case of matter on the branes will be given afterwards.

5.3.6 Relaxing the approximation

Gauge Invariance

The previous result has been derived assuming that A in (5.27) was independent of y. It is possible to show that such a dependence does not affect the final result. In order to show this, we will separate the y-derivatives of A from the other y-derivatives. Then by summing over the derivatives of A, we will recover the quantity d. We denote by $\delta_A = A'(y)\partial_A + A''(y)\partial_{A'} + \partial_{\alpha}A'(y)\partial_{\partial_{\alpha}A} + \cdots$ the y-derivative which acts exclusively on A. In order to differentiate any other quantity, we can apply the operator $\bar{\partial}_y = \partial_y - \delta_A$ which has no effect on A. In particular we write:

$$\bar{\partial}_y Q(y) = \partial_y Q(y) \equiv Q'(y), \tag{5.62}$$

for any quantity $Q = K_{\nu}^{\mu}, E_{\nu}^{\mu}, q_{\mu\nu}, \Gamma_{\mu\nu}^{\alpha}$. In the previous section, A was y-independent and so the operator δ_A had no effect, we simply had $\bar{\partial}_y = \partial_y$. Now this relation does not hold anymore and we need to include the effect of δ_A in the Taylor expansion (5.32) bearing in mind that δ_A and $\bar{\partial}_y$ do not commute. Indeed, a new A appears each time the operator $\bar{\partial}_y$ is applied on a quantity Q since $\bar{\partial}_y Q = A\mathcal{L}_n Q$, where \mathcal{L}_n is the Lie derivative along the normal direction. So $\delta_A Q = 0$ whereas $\delta_A \bar{\partial}_y Q = \hat{U}A'(y)$ where \hat{U} is some four-dimensional operator. If we consider for instance the y-derivative of the extrinsic curvature in (5.30), we have:

$$\delta_A \bar{\partial}_y K^{\mu}_{\nu}(y) = \delta_A K^{\mu'}_{\nu}(y)$$

= $-A'(y) E^{\mu}_{\nu}(y) - D^{\mu} D_{\nu} A'(y) - A'(y) K^{\mu}_{\alpha}(y) K^{\alpha}_{\nu}(y) + A'(y) \frac{1}{L^2} \delta^{\mu}_{\nu}.$ (5.63)

In the Taylor expansion, (5.32), there will be a term, which we denote by $\mathscr{K}_{1\nu}^{\mu}(0)$, that has only the first derivative along $\bar{\partial}_y$ and all the other ones along δ_A :

$$\mathscr{K}_{1\ \nu}^{\mu}(y) = \sum_{m \ge 1} \frac{1}{m!} \delta_A^{(m-1)} \bar{\partial}_y K_{\nu}^{\mu}(y).$$
(5.64)

We may write

$$\delta_A^{(m-1)} \bar{\partial}_y K^{\mu}_{\nu}(y) = \hat{U}^{\mu}_{\nu} A^{(m-1)}, \qquad (5.65)$$

where the operator \hat{U}^{μ}_{ν} can be read off from (5.63). In particular, on the empty positive brane,

$$\hat{U}^{\mu}_{\nu}(y=0) = -E^{\mu}_{\nu}(y) - D^{\mu}D_{\nu}|_{y=0}.$$
(5.66)

Using this relation, we have:

$$\mathcal{K}_{1 \nu}^{\mu}(y) = \hat{U}_{\nu}^{\mu} \sum_{m \ge 1} \frac{1}{m!} A^{(m-1)}(y) = \hat{U}_{\nu}^{\mu} \int_{y}^{y+1} A(y') \, \mathrm{d}y'.$$
(5.67)

In particular the term $\mathscr{K}_{1\nu}^{\mu}(0)$ that contributes to the Taylor expansion (5.32) is:

$$\mathscr{K}^{\mu}_{1\nu}(0) = \hat{U}^{\mu}_{\nu}(y=0) \int_{0}^{1} A(y) \,\mathrm{d}y$$
(5.68)

$$= -d E^{\mu}_{\nu}(0) - D^{\mu} D_{\nu} d|_{y=0}, \qquad (5.69)$$

which is precisely the first term (5.39) that was contributing to the Taylor expansion when A was assumed to be *y*-independent. In what follows we shall see that the Taylor expansion can be written as a sum of \mathscr{K}_n which expression is precisely the same as $\mathcal{K}^{(n)}$ when A is assumed to be *y*-independent.

We shall work in a symbolic way, omitting any indices. First we notice that the Taylor expansion (5.32) or (5.38) can be written in the form:

$$\sum_{n \ge 1} \frac{1}{n!} \mathscr{K}_n(0) = 0, \tag{5.70}$$

with

$$\frac{1}{n!}\mathscr{K}_{n}(y) = \sum_{m_{1},\cdots,m_{n}\geq 0} \frac{1}{l_{n}!} \mathcal{F}_{n}^{m_{1},\cdots,m_{n}}, \qquad (5.71)$$

where we wrote $l_n = (m_1 + \cdots + m_n + n)$ and

$$\mathcal{F}_{n}^{m_{1},\cdots,m_{n}} = \delta_{A}^{(m_{n})} \bar{\partial}_{y} \delta_{A}^{(m_{n-1})} \bar{\partial}_{y} \cdots \bar{\partial}_{y} \delta_{A}^{(m_{1})} \bar{\partial}_{y} K.$$
(5.72)

Recalling that each time K is differentiated with respect to y, a new power of A appears, we therefore have:

$$\bar{\partial}_{y}^{(n)}K = \hat{U}_{n} \left[A\hat{U}_{n-1} \left(\cdots \hat{U}_{1}A \right) \right], \qquad (5.73)$$

where each operator \hat{U}_i depends on y but includes only derivatives along the y = consthypersurface and quantities $Q = K, E, q, \Gamma$ *ie.* \hat{U}_i does not depend on A. Using this notation, \mathcal{F}_n is:

$$\mathcal{F}_{n}^{m_{1},\cdots,m_{n}} = \hat{U}_{n}\,\delta_{A}^{(m_{n})}\,\left[A\hat{U}_{n-1}\,\delta_{A}^{(m_{n-1})}\,\left(A\cdots\hat{U}_{2}\,\delta_{A}^{(m_{2})}\,\left(A\hat{U}_{1}A^{(m_{1})}\right)\cdots\right)\right],\qquad(5.74)$$

where the operators \hat{U}_i should be treated as independent variable of y so that for instance

$$\delta_A^{(m_2)} \left(A \hat{U}_1 A^{(m_1)} \right) = \sum_{k \ge 0} C_{m_2}^k A^{(k)} \hat{U}_1 A^{(m_1 + m_2 - k)}.$$
(5.75)

When all the sums are performed, we obtain:

$$\frac{1}{n!} \mathscr{K}_n(y) = \hat{U}_1 \int \left[A \hat{U}_2 \left(A \int \cdots \int \left(A \hat{U}_n \int A dy \right) dy \cdots \right) dy \right] dy.$$
(5.76)

So on the brane, we can express \mathcal{K}_n in the form:

$$\mathscr{K}_n(0) = n! V_1(1) \tag{5.77}$$

$$V_m(y) = \hat{U}_m \int_0^y A(y) U_{m+1}(y) dy, \quad m < n$$
(5.78)

$$V_n(y) = \hat{U}_n \int_0^y A(y) dy = \hat{U}_n \,\tilde{d}(y), \qquad (5.79)$$

with $\tilde{d}(y,x) = \int_0^y A(y) dy$. So far this result is exact. When the branes are close, this result will simplify remarkably. First we point out that $0 < \tilde{d}(y) < d \ll L$ for any 0 < y < 1. This

follows from the fact that the bulk geometry is completely regular, so between the branes, A(y,x) > 0, which implies $0 < \int_0^y A(y,x) dy < d(x) \ll L$ for any 0 < y < 1. This holds similarly for any multiple integral. The same simplifications as in the previous section will therefore be valid here. For instance if in the previous section we had $\hat{U}_1 d \simeq \partial d$, then this will remain true in this case as well: $\hat{U}_1 \tilde{d}(y,x) \simeq \partial \tilde{d}(y,x)$.

In the following we shall label with a "bar" any quantity which was derived in the previous section assuming A was independent of y. In the previous section, we had $\bar{A} = d$ and so V_m was simply:

$$\bar{V}_m(y) = \frac{y^{n+1-m}}{(n+1-m)!} \,\hat{U}_m\left[d\,\hat{U}_{m+1}\left(d\,\hat{U}_{m+2}\cdots d\,\hat{U}_n d\right)\right],\tag{5.80}$$

so we had:

$$\bar{K}^{(n)}(0) = \hat{U}_1 \left[d \, \hat{U}_2 \left(d \, \hat{U}_3 \cdots d \, \hat{U}_n d \right) \right], \tag{5.81}$$

using the fact that at $d \ll L$, the action of the operators \hat{U}_i was considerably simplified and at leading order in d/L, they were equivalent to the overall operator $\hat{U}^{(n)}$:

$$\bar{K}^{(n)}(0) = \hat{U}^{(n)} d^n \tag{5.82}$$

with
$$\hat{U}^{(2n+1)} = \frac{1}{(2n+1)!} \left(-E(0) - D^2\right) D^{(2n)}$$
 (5.83)

$$\hat{U}^{(2n)} = \frac{1}{2n!} K(0) D^{(2n)}, \qquad (5.84)$$

where D is a four-dimensional covariant derivative along the y = const hypersurface. The leading contribution was indeed:

$$\bar{K}^{(2n+1)}(0) = K'(0) (\partial d)^{2n}$$
(5.85)

$$\bar{K}^{(2n)}(0) = K(0) (\partial d)^{2n}.$$
 (5.86)

Now we may go back to the situation where A has some y-dependence. Because $0 < \tilde{d}(y,x) < d(x) \ll L$ for any 0 < y < 1 and similarly for any multiple integral in V_m , when the operators \hat{U}_i are applied on these integrals, we can reproduce step by step exactly the same procedure as we followed in the previous section in order to keep only the leading

order in d/L. In particular, the repeated action of each \hat{U}_i on each multiple integral can be substituted, in the small d limit (which implies a small $\int_0^y \cdots \int_0^y A dy$ limit) by the action of an overall operator $\hat{U}^{(n)}$. If we do so, the leading contribution can be expressed in the same way as in (5.82):

$$\frac{1}{n!}\mathscr{K}_n(0) = \hat{U}^{(n)} \left[\int_0^1 A\left(\int_0^y \cdots A \int_0^y A \mathrm{d}y \right) \right], \qquad (5.87)$$

with the operator $\hat{U}^{(n)}$ as given in (5.83, 5.84). Now the multiple integral is simply:

$$\int_0^1 A\left(\int_0^y \cdots A\int_0^y A \mathrm{d}y\right) = \frac{1}{n!} \left(\int_0^1 A \mathrm{d}y\right)^n \equiv \frac{d^n}{n!}.$$
(5.88)

So the rest follows exactly as in the previous case. For any n, the leading contribution to \mathscr{K}_n is:

$$\mathscr{K}_n(0) = \hat{U}^{(n)} d^n, \qquad (5.89)$$

exactly as in (5.82). So the leading order in d/L of the Taylor expansion (5.38) is independent on the y-dependence of A, and in particular we obtain the same result whether A = d or any other function of y such that $\int_0^1 A \, dy = d$. The result derived in the previous section is therefore valid independently of the precise expression of A, as this should be since our result is gauge invariant. This result comes without surprise but had to be checked for consistency of the close-brane formalism developed in the previous section.

The other assumption that was made to derive the result of section 5.3.5 was taking the branes to be empty. This assumption was only made to simplify the notation and to focus on the main ideas, however all the procedure is completely generalisable to the case where matter is present on the brane as we shall in the next subsection.

Matter on the brane

We can now give a brief overview of what would happen when matter is introduced on the branes. The exact, expression of the extrinsic curvature on the branes is modified to (2.17)

$$K^{\mu}_{\nu}(y=0) = -\frac{1}{L} \,\delta^{\mu}_{\nu} + \frac{\kappa}{2} \,\tilde{T}^{(+)\,\mu}_{\ \nu} \tag{5.90}$$

$$K^{\mu}_{\nu}(y=1) = -\frac{1}{L} \,\delta^{\mu}_{\nu} - \frac{\kappa}{2} \,\tilde{T}^{(-)\,\mu}_{\ \nu} \tag{5.91}$$

where we use the same notation as in section (2.8.2) $\tilde{T}^{(\pm)\,\mu}_{\ \nu} = T^{(\pm)\,\mu}_{\ \nu} - \frac{1}{3}T^{(\pm)}\delta^{\mu}_{\nu}$. However, despite this modification, the method remains the same. In particular $\mathcal{K}^{(m)} \sim d^0$ for both m = 1 and m = 2, and using the same recursive argument, we may show that this remains true for any $m \ge 0$. The symbolic expression (5.44) for the second derivative is still valid. So the leading contribution still comes from the derivative of the Christoffel symbol, which formal derivative can still be expressed the same way. To summarise, we therefore still have the following schematic steps which are completely valid even in the presence of matter:

$$\begin{aligned}
\mathcal{K}_{\nu}^{\mu\prime} &= \underbrace{-D^{\mu}D_{\nu}d}_{\nu} + \cdots \\
\mathcal{K}_{\nu}^{\mu\prime\prime} &= \underbrace{q^{\beta\mu}\Gamma_{\beta\nu}^{\alpha\prime}\partial_{\alpha}d}_{\rho} + \mathcal{O}(d/L) \\
&= & \hat{O}\underbrace{K_{\nu}^{\mu}}_{\nu} + \mathcal{O}(d/L) \\
\mathcal{K}_{\nu}^{\mu(3)} &= & \hat{O}\mathcal{K}_{\nu}^{\mu\prime} \\
&\vdots \\
\mathcal{K}_{\nu}^{\mu(n)} &= & \hat{O}\mathcal{K}_{\nu}^{\mu(n-2)}.
\end{aligned}$$

The main result that $\mathcal{K}^{\mu(n)}_{\nu} = \hat{O}\mathcal{K}^{\mu(n-2)}_{\nu}$ can therefore be completely generalised to the situation when matter is present on the branes.

5.3.7 Normal Derivative of the Extrinsic Curvature on the brane

We now may use these results to perform the sum of the Taylor expansion (5.32) which can be expressed as (5.36) in the small *d* limit. Using the notation of the previous subsection and the fact that $\mathcal{K}^{\mu(n)}_{\nu} = \hat{O}\mathcal{K}^{\mu(n-2)}_{\nu}$, the Taylor can be expressed as

$$\sum_{n=0}^{\infty} \frac{1}{n!} \mathcal{K}_{\nu}^{\mu(n)} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \hat{O}^{n} \mathcal{K}_{\nu}^{\mu(1)} + \frac{1}{(2n)!} \hat{O}^{n} \mathcal{K}_{\nu}^{\mu(0)}$$
$$= \frac{\sinh \sqrt{\hat{O}}}{\sqrt{\hat{O}}} \mathcal{K}_{\nu}^{\mu(1)} + \cosh \sqrt{\hat{O}} \mathcal{K}_{\nu}^{\mu(0)}.$$
(5.92)

We then have a formal expression for the first derivative of the extrinsic curvature on the brane:

$$\frac{\partial}{\partial y}K^{\mu}_{\nu}\Big|_{y=0} = -\frac{\sqrt{\hat{O}}}{\sinh\sqrt{\hat{O}}}\Big[\cosh\sqrt{\hat{O}}\,K^{(+)\,\mu}_{\ \nu} - K^{(-)\,\mu}_{\ \nu}\Big],\tag{5.93}$$

where $K^{(\pm)\mu}_{\ \nu} = K^{\mu}_{\nu}(y=0,1)$. The same method could be repeated using the negativetension brane as the reference brane instead. This would be equivalent to taking -d instead of d all the way through. The derivative of the extrinsic curvature on the negative-tension brane is therefore

$$\frac{\partial}{\partial y}K^{\mu}_{\nu}\Big|_{y=1} = \frac{\sqrt{\hat{O}}}{\sinh\sqrt{\hat{O}}}\Big[\cosh\sqrt{\hat{O}}K^{(-)\,\mu}_{\ \nu} - K^{(+)\,\mu}_{\ \nu}\Big].$$
(5.94)

We can now implement what is known from the Israël matching conditions on the brane: the extrinsic curvatures of the two branes are given in terms of the stress-energy tensors and the tensions by the expressions (5.90) and (5.91). The derivative of the extrinsic curvature on the branes can therefore be written as

$$\partial_y K^{(\pm)\,\mu}_{\ \nu} = \pm \sqrt{\hat{O}} \, \frac{\cosh\sqrt{\hat{O}} - 1}{\sinh\sqrt{\hat{O}}} \, \delta^{\mu}_{\nu} + \frac{\kappa}{2} \frac{\sqrt{\hat{O}}}{\sinh\sqrt{\hat{O}}} \, \left(\cosh\sqrt{\hat{O}} \, \tilde{T}^{(\pm)\,\mu}_{\ \nu} + \tilde{T}^{(\mp)\,\mu}_{\ \nu}\right). \tag{5.95}$$

Using the fact that

$$\hat{O}\delta^{\mu}_{\nu} \equiv F^{\mu}_{\nu} = 2\partial^{\mu}d\partial_{\nu}d - (\partial d)^{2}\delta^{\mu}_{\nu}$$
(5.96)

$$\hat{O}F^{\mu}_{\nu} = (\partial d)^4 \,\delta^{\mu}_{\nu}, \qquad (5.97)$$

the first term in (5.95) can be expressed exactly as

$$\sqrt{\hat{O}} \frac{\cosh\sqrt{\hat{O}} - 1}{\sinh\sqrt{\hat{O}}} \delta^{\mu}_{\nu} = \frac{1}{2L} \left[|\partial d| \left(\tanh\frac{|\partial d|}{2} - \tan\frac{|\partial d|}{2} \right) \delta^{\mu}_{\nu} + \frac{1}{|\partial d|} \left(\tanh\frac{|\partial d|}{2} + \tan\frac{|\partial d|}{2} \right) F^{\mu}_{\nu} \right]$$

$$\text{with } |\partial d| \equiv \sqrt{-q^{(+)\,\mu\nu}\partial_{\mu}d\partial_{\nu}d}.$$
(5.98)
(5.99)

We may point out that this procedure seems to break down in the limit where $|\partial d| \to \pi$, for which $\tan \frac{|\partial d|}{2} \to \infty$. Thus the action of the operator $\sqrt{\hat{O}} \frac{\cosh \sqrt{\hat{O}}-1}{\sinh \sqrt{\hat{O}}}$ seems to be ill-defined in that case. This might be an interesting feature which has not been pointed out before, however we do not believe that this limit has much physical relevance or that anything happens when the fifth dimension expansion approaches to this value. In what follows we will check that when physical quantities are calculated, the relevant part does not depend on the trigonometric quantities but rather on their hyperbolic counterpart which have no singularities at $|\partial d| = \pi$.

The action of the operator \hat{O} in a generic stress-energy tensor is, on the other hand, less trivial, and a generic expression in the covariant form can not be given in general. However we will see in what follows that for the specific cosmological case we shall be interested in, the second part of (5.95) can be simplified as well. But first we can look at the effective close-brane theory that we find on the brane.

5.4 Close-brane effective theory

5.4.1 Four-dimensional Close-brane Effective Theory

In the previous section we have shown that an effective theory could be derived on the brane, in the close-brane regime. We can now put all the results together and show explicitly what this theory is and test it in different regimes.

Having an expression for both the extrinsic curvature on the branes (5.90, 5.91) and its derivative (5.95), the Gauss equation (5.31) is now completely determined and gives rise to the modified Einstein equations on the branes:

$$\begin{aligned}
G^{(\pm)\mu}{}_{\nu} &\approx \frac{1}{d} \Big[D^{(\pm)\mu} D^{(\pm)}_{\nu} d \pm \frac{|\partial d|}{L} \tanh \frac{|\partial d|}{2} \delta^{\mu}_{\nu} \\
&\pm \frac{1}{L |\partial d|} \left(\tanh \frac{|\partial d|}{2} + \tan \frac{|\partial d|}{2} \right) \partial^{\mu} d \partial_{\nu} d \\
&+ \frac{\kappa}{2} \frac{\sqrt{\hat{O}}}{\sinh \sqrt{\hat{O}}} \left(\cosh \sqrt{\hat{O}} \tilde{T}^{(\pm)\mu}{}_{\nu} + \tilde{T}^{(\mp)\mu}{}_{\nu} \right) \Big].
\end{aligned} \tag{5.100}$$

Furthermore, from the tracelessness of the Weyl tensor, the Ricci scalar in (5.29) satisfies

$$R^{(\pm)} = \mp \frac{\kappa}{L} T^{(\pm)} + \frac{\kappa^2}{4} \left(\tilde{T}^{(\pm)\,2} - \tilde{T}^{(\pm)\,\alpha}_{\ \beta} \tilde{T}^{(\pm)\,\beta}_{\ \alpha} \right),$$
(5.101)

and the Klein Gordon equation for the scalar field is constrained to be

$$\Box^{(\pm)}d \approx \mp \frac{|\partial d|}{L} \left(3\tanh\frac{|\partial d|}{2} - \tan\frac{|\partial d|}{2}\right) + \frac{\kappa}{6} \frac{\sqrt{\hat{O}}}{\sinh\sqrt{\hat{O}}} \left(\cosh\sqrt{\hat{O}} T^{(\pm)} + T^{(\mp)}\right).(5.102)$$

Furthermore, the conservation of the stress-energy tensor on the branes follows directly from the Codacci equation (2.8)

$$D^{(\pm)}_{\mu}T^{(\pm)\,\mu}_{\ \nu} = 0. \tag{5.103}$$

This procedure has allowed us to derive an effective close-brane theory on both the positiveand negative-tension branes in a covariant way. These expressions are exact in the closebrane limit and the formalism could be generalised to work out the effective four-dimensional Einstein tensor on any y = const hypersurface.

The effective theory on the branes describes gravity coupled to a scalar field in a nontrivial way. This theory has some higher-derivative corrections as expected but, remarkably, they are both simple in form and entirely *first order*, *ie.* involving only powers of first derivatives. This leads to the important result that the theory remains second order in derivatives even when correct to *all* orders in velocity. The initial data on a Cauchy surface that needs to be specified for the theory to be solved is the same as that needed in the lowenergy effective theory. There will be no need to specify extra information or to consider the corrections to be small, a feature which will make the theory straightforward to solve. Although this theory does not seem to be derivable from a four-dimensional action, it may be solved using standard four-dimensional methods.

5.4.2 Conservation of Energy

In order for the theory to be consistent, the Bianchi identity on the brane needs to be preserved. This means that, to leading order in d/L, the Klein-Gordon equation (5.102) must imply the right hand side of (5.100) to be transverse. In other words, if we consider the right hand side of (5.100) to be the stress-energy tensor $T_{\mu\nu}^{\text{eff}}$ for the scalar field d and matter on the brane, the Klein Gordon condition should impose that it is conserved, *ie.* divergenceless. In order to check this requirement, we will focus on the positive tension brane. We will see that it is important to add the next to leading correction $d\bar{E}_{\mu\nu}$ where $\bar{E}_{\mu\nu} \sim d^0$. As we shall see, although $\bar{E}_{\mu\nu}$ does not contribute to the leading order of the Einstein equation, it does contribute to the leading order of the divergence of $T_{\mu\nu}^{\text{eff}}$. This is due to the fact that, although $d\bar{E}_{\mu\nu} \sim d$ is negligible, its derivative is not $(\partial_{\mu}d) \bar{E}^{\mu}_{\nu} \sim d^{0}$. We therefore need to consider its contribution as well. The transverse requirement will therefore impose a condition on the next to leading order in d.

Using the equation of motion for the scalar field, we may rewrite (5.100) and (5.102) in the form:

$$\begin{aligned}
G_{\mu\nu}^{(+)} &= T_{\mu\nu}^{(+)\,\text{eff}} \\
T_{\mu\nu}^{(+)\,\text{eff}} &= \frac{1}{d} \Big[D_{\mu} D_{\nu} d + \frac{2}{L} g \,\partial_{\mu} d\partial_{\nu} d - d \,\bar{E}_{\mu\nu} - \left(\Box \, d + \frac{z^2 g}{L} + \frac{3f}{L} - d \,\bar{E} \right) q_{\mu\nu}^{(+)} \Big] (5.104) \\
&+ \frac{\kappa}{2d} \frac{\sqrt{\hat{O}}}{\sinh\sqrt{\hat{O}}} \left(\cosh\sqrt{\hat{O}} \, T^{(+)\,\mu}_{\ \nu} + T^{(-)\,\mu}_{\ \nu} \right) \\
\Box^{(+)} d &= -\frac{1}{L} \left(4f + 2z^2 g \right) + d \,\bar{E} + \frac{\kappa}{6} \frac{\sqrt{\hat{O}}}{\sinh\sqrt{\hat{O}}} \left(\cosh\sqrt{\hat{O}} \, T^{(+)} + T^{(-)} \right), \quad (5.105)
\end{aligned}$$

where for simplicity we used the notation $z = |\partial d|$, $f \equiv f(z^4) = \frac{z}{2} \left(\tanh \frac{z}{2} - \tan \frac{z}{2} \right)$ and $g \equiv g(z^4) = \frac{1}{2z} \left(\tanh \frac{z}{2} + \tan \frac{z}{2} \right)$. f and g are viewed as functions of $z^4 = (\partial d)^4$ since their Taylor expansions only contain powers of z^4 . Therefore, for example,

$$\partial_{\mu}f(z) = 2z^2 f'(z) \,\partial_{\mu}\left(z^2\right). \tag{5.106}$$

The divergence of the stress-energy tensor as defined in (5.104) is given by:

$$dD_{\mu}T^{(+)\,\text{eff}\,\mu}_{\nu} = -\frac{12}{L^2} \left[gf + z^2 \left(f - z^2 g \right) \left(f' + z^2 g' \right) \right] \partial_{\nu}d - \left[\bar{E}^{\mu}_{\nu} - \bar{E}\delta^{\mu}_{\nu} \right] \partial_{\mu}d + \frac{1}{L^2} \mathcal{O}(d/L).$$
(5.107)

For the stress-energy to be transverse to leading order in d, we therefore have a constraint equation on the next-to-leading order contribution $\bar{E}_{\mu\nu}$:

$$\left[\bar{E}^{\mu}_{\nu} - \bar{E}\delta^{\mu}_{\nu}\right]\partial_{\mu}d = -\frac{12}{L^{2}}\left[gf + z^{2}\left(f - z^{2}g\right)\left(f' + z^{2}g'\right)\right]\partial_{\nu}d.$$
(5.108)

Furthermore, from the eq.(2.45), at low-energy and to leading order in velocities and in d/L, $\bar{E}_{\mu\nu}$ should be:

$$\bar{E}_{\mu\nu} = \frac{1}{L} \left[D_{\mu} D_{\nu} d + \frac{1}{L} \left(\partial_{\mu} d \partial_{\nu} d - \frac{1}{2} \left(\partial d \right)^2 q_{\mu\nu}^{(+)} \right) \right] + \frac{1}{L^2} \mathcal{O} \left(\left(\partial d \right)^4, d/L \right).$$
(5.109)

A natural ansatz for $\bar{E}_{\mu\nu}$ is therefore:

$$\bar{E}_{\mu\nu} = \frac{1}{L} \left[D_{\mu} D_{\nu} d + \frac{1}{L} \left(\bar{f}(z^4) q_{\mu\nu}^{(+)} + \bar{g}(z^4) F_{\mu\nu} \right) \right].$$
(5.110)

Using this ansatz in the constraint (5.108), we find the equations for \bar{f} and \bar{g} :

$$\bar{f} - f + z^2(\bar{g} - g) = 4(gf + z^4fg' - z^4gf') + 4z^2(ff' - z^4gg').$$

Since \bar{f} and \bar{g} are functions of z^4 uniquely, their expression is:

$$\bar{f}(z^4) = f + 4gf + 4z^4 \left(fg' - gf'\right) \tag{5.111}$$

$$\bar{g}(z^4) = g + 4ff' - 4z^4gg'. \tag{5.112}$$

In the slow-velocity limit, we may check that $\bar{f} = \mathcal{O}((\partial d)^4)$ and $\bar{g} = \frac{1}{2} + \mathcal{O}((\partial d)^4)$ so we recover the result from the low-energy effective theory (5.109). Furthermore, using the expression (2.46), the next to leading order $\bar{E}_{\mu\nu}$ (5.110) will be conformally related to the leading order: $\bar{E}_{\mu\nu} = (d/L) E^0_{\mu\nu}$ only if $\bar{f} = f$ and $\bar{g} = g$. However, from the expressions (5.111, 5.112), this can only be the case in the low-velocity limit where $\bar{f} = f = -\frac{z^4}{4!} + \mathcal{O}(z^8)$ and $\bar{g} = g = \frac{1}{2} + \mathcal{O}(z^4)$, but the higher order terms in velocities do not match and hence the conformal relation $\bar{E}_{\mu\nu} = (d/L) E^0_{\mu\nu}$ does not hold.

This expression for $E_{\mu\nu}$ with the relations (5.111, 5.112) is consistent with the close limit theory since it ensures that the stress-energy tensor is conserved to leading order in d/L. However we may emphasise that this is not the only possible answer and relies strongly on the ansatz (5.110). Since we only want to focus in the leading order in d/Lof the theory, the exact expression of $\bar{E}_{\mu\nu}$ will not be relevant, it is enough to show that it is possible to introduce its contribution in such a way that the transverse condition is preserved. If we wanted to go further in the study of the next to leading order term of the theory, we should check that this expression of $\bar{E}_{\mu\nu}$ is consistent with its exact expression that can be derived in the background using the SAdS five-dimensional geometry. This is beyond the purpose of this chapter which aim is to focus on the close-brane limit and to neglect any terms which are not of leading order in the expansion.

5.4.3 Tests on the Close-brane Effective Theory

Low-energy limit

As a first consistency check of this close-brane theory, we consider its low-energy limit and compare it with the effective four-dimensional low-energy theory section 2.6 [95, 109, 113, 114] for small brane separation. In that common limit, both theories should agree.

In the low-energy limit, the magnitude of the stress-energy tensor on the brane is small compared to the brane tension. Any quadratic term $\kappa^2 T^{(\pm)2}$ is negligible compared to $\kappa/L T^{(\pm)}$, so the quadratic terms may be dropped in (5.101). The Ricci tensor on the brane is then

$$R^{(\pm)} = \mp \frac{\kappa}{L} T^{(\pm)}.$$
 (5.113)

Furthermore in the low-energy limit, the branes are moving slowly, $|\partial d| \ll 1$, to linear order in $|\partial d|$, we may use $\sinh \sqrt{\hat{O}} \sim \sqrt{\hat{O}}$ and $\cosh \sqrt{\hat{O}} \sim 1$, or equivalently, $\tan \frac{|\partial d|}{2} \simeq \tanh \frac{|\partial d|}{2} \simeq \frac{1}{2} |\partial d|$ in (5.100) and (5.102). The effective Einstein equation on the brane at low energy is therefore

$$G^{(\pm)\mu}_{\ \nu} \approx \frac{1}{d} D^{(\pm)\mu} D^{(\pm)}_{\nu} d + \frac{\kappa}{2d} \left(\tilde{T}^{(+)\mu}_{\ \nu} + \tilde{T}^{(-)\mu}_{\ \nu} \right) \\
 \pm \frac{1}{Ld} \left(\partial^{\mu} d \, \partial_{\nu} d - \frac{1}{2} \left(\partial d \right)^{2} \delta^{\mu}_{\nu} \right),
 \tag{5.114}$$

with the equation of motion for d:

$$\Box^{(\pm)}d = \pm \frac{1}{L} \left(\partial d\right)^2 + \frac{\kappa}{6} \left(T^{(+)} + T^{(-)}\right).$$
(5.115)

We can therefore write (5.114) in the more common form:

$$G^{(\pm)\mu}{}_{\nu} \approx \frac{1}{d} \left(D^{(\pm)\mu} D^{(\pm)}_{\nu} d - \Box^{(\pm)} d \,\delta^{\mu}_{\nu} \right) \pm \frac{1}{Ld} \left(\partial^{\mu} d \,\partial_{\nu} d + \frac{1}{2} \left(\partial d \right)^{2} \delta^{\mu}_{\nu} \right) \qquad (5.116) \\
 + \frac{\kappa}{2d} \left(T^{(+)\mu}{}_{\nu} + T^{(-)\mu}{}_{\nu} \right),$$

which is precisely the result in (2.45), (2.48), (2.49) that we get from the effective lowenergy theory in the close-brane limit. We might think that the low-energy effective theory (2.45) could still give a correct answer provided that the relation between the conformal factor Ψ and the proper distance between the branes d was modified. If that was the case, the relation between these two variables should include some higher order velocity terms: $\Psi = \Psi(d, \partial d)$ such that, in the low-velocity limit, $\Psi(d, \partial d) \sim e^{-d/L}$. The theory should then include some terms of higher order in derivative of the form $\partial^{\alpha} dD_{\mu}D_{\nu}\partial_{\alpha} d\Psi_{,\partial d}$. This is not compatible with the theory in (2.45) which only contains terms up to second order in derivatives. The close limit theory in (5.100) and the low-energy effective theory (2.45) are therefore genuinely different.

Cosmological Symmetry

Having a theory which should be exact in the close-brane limit, we may now check that it correctly reproduces the exact background behaviour obtained in (5.12). As already mentioned in section 5.3.7, in a general case, the exact coupling to matter can only be expressed in a formal way in terms of the action of the operator \hat{O} on the stress-energy tensors. However, if we are concerned with the specific case of cosmological symmetry, as a check on the validity of the close-brane theory, we will see that the second part of (5.95) can be simplified and the coupling to matter is indeed tractable.

For simplicity we shall consider flat spatial dimensions. First we point out that the tracelessness of the Weyl tensor forces the Ricci tensor to be given by (5.101). For the background, this implies the condition on the Hubble parameter:

$$R^{(+)} = 6\left(\dot{H}_{+} + 2H_{+}^{2}\right) = -\frac{\kappa}{L}T^{(+)} + \frac{\kappa^{2}}{4}\left(\tilde{T}^{(+)\,2} - \tilde{T}^{(+)\,\alpha}_{\ \beta}\tilde{T}^{(+)\,\beta}_{\ \alpha}\right)$$
$$= -\frac{\kappa}{L}\left(-\rho_{+} + 3p_{+}\right) - \frac{\kappa^{2}}{6}\rho_{+}\left(\rho_{+} + 3p_{+}\right), \qquad (5.117)$$

where ρ_+ is the energy density of the matter fields living on the positive brane, and p_+ is the pressure. Using the same notation as in section 5.2, H_+ is the Hubble parameter on the positive brane and \dot{H}_+ its derivative with respect to the proper time. Using this result together with the conservation of energy condition (5.103): $\dot{\rho}_+ = -3(\rho_+ + p_+)H_+$, we may solve this differential equation for H_+ . The Hubble parameter on the brane is then

$$H_{+}^{2} = \frac{C}{L^{2}a_{+}^{4}} + \frac{\kappa}{3L}\rho_{+} + \frac{\kappa^{2}}{36}\rho_{+}^{2}, \qquad (5.118)$$

where C is an integration constant. This is in agreement with (2.28). In what follows we shall see that the integration constant C is related to the radion degree of freedom in exactly the same way as in the exact five-dimensional theory. For this we will show that in the close-brane limit, the relation (5.12) for the Hubble parameter in terms of the radion dynamics is satisfied.

For the background, any tensor can be expressed in the form:

$$Z^{\mu}_{\nu} = \begin{pmatrix} Z^{0}_{0} & 0\\ 0 & Z^{i}_{j} \end{pmatrix}, \qquad (5.119)$$

where $Z_j^i \sim \delta_j^i$. Working in terms of the proper time, $|\partial d| = \dot{d}$, the effect of the operator \hat{O} on such a tensor is:

$$\hat{O}Z^{\mu}_{\nu} = \dot{d}^2 \,\tilde{Z}^{\mu}_{\nu} \equiv \dot{d}^2 \begin{pmatrix} -Z^0_0 & 0\\ 0 & Z^i_j \end{pmatrix}.$$
(5.120)

Using this notation, the derivative of the extrinsic curvature on the positive tension brane in (5.93), may be expressed as

$$\partial_{y}K^{(+)\mu}{}_{\nu} = -\frac{\sqrt{\hat{O}}}{\sinh\sqrt{\hat{O}}} \Big[\cosh\sqrt{\hat{O}} K^{(+)\mu}{}_{\nu} - K^{(-)\mu}{}_{\nu} \Big] \\ = -\frac{1}{2} \left(\frac{\cosh\dot{d}}{\sinh\dot{d}} + \frac{\cos\dot{d}}{\sin\dot{d}} \right) K^{(+)\mu}{}_{\nu} - \frac{1}{2} \left(\frac{\cosh\dot{d}}{\sinh\dot{d}} - \frac{\cos\dot{d}}{\sin\dot{d}} \right) \tilde{K}^{(+)\mu}{}_{\nu} \quad (5.121) \\ + \frac{1}{2} \left(\frac{1}{\sinh\dot{d}} + \frac{1}{\sin\dot{d}} \right) K^{(-)\mu}{}_{\nu} + \frac{1}{2} \left(\frac{1}{\sinh\dot{d}} - \frac{1}{\sin\dot{d}} \right) \tilde{K}^{(-)\mu}{}_{\nu},$$

we therefore have:

$$\partial_y K^{(+)\,0}_{\ 0} = \frac{\dot{d}}{\sin \dot{d}} \left(-\cos \dot{d} \ K^{(+)\,0}_{\ 0} + K^{(-)\,0}_{\ 0} \right) \tag{5.122}$$

$$\partial_y K^{(+)\,i}_{\ j} = \frac{d}{\sinh \dot{d}} \left(-\cosh \dot{d} \ K^{(+)\,i}_{\ j} + K^{(-)\,i}_{\ j} \right), \tag{5.123}$$

which can simply substitute in the modified Einstein equation (5.31)

$$G^{(+)\,\mu}_{\ \nu} = \frac{1}{d} \Big[D^{\mu} D_{\nu} d + \partial_{y} K^{(+)\,\mu}_{\ \nu} + \mathcal{O}(d/L) \Big].$$
(5.124)

In particular for the *ii*-component of this equation, we have

$$G_{1}^{(+)1} = H_{+}^{2} - \frac{1}{3}R^{(+)}$$
(5.125)

$$= \frac{\dot{d}}{d} \left[-H_{+} - \frac{\cosh \dot{d}}{\sinh \dot{d}} K^{(+)}{}_{1}^{1} + \frac{1}{\sinh \dot{d}} K^{(-)}{}_{1}^{1} \right], \qquad (5.126)$$

where the first line is simply the expression of the Einstein tensor for the flat FRW geometry and the second line is a consequence of (5.124), using $D^1D_1 d = -\dot{d} H_+$. To leading order in d/L the Hubble parameter should satisfy

$$H_{+} \approx -\frac{\cosh d}{\sinh \dot{d}} K^{(+)}{}_{1}^{1} + \frac{1}{\sinh \dot{d}} K^{(-)}{}_{1}^{1}, \qquad (5.127)$$

unless \dot{d} goes to zero at the collision, which is not the situation we are interested in, as discussed in section 5.3.2. Recalling from the Israël Matching condition (5.90), (5.91) that

$$K^{(\pm)\,i}_{\ \ j} = \left(-\frac{1}{L} \pm \frac{\kappa}{6}\rho_{\pm}\right)\delta^i_j,\tag{5.128}$$

we therefore have:

$$H_{+} \approx \frac{1}{L} \tanh \frac{\dot{d}}{2} - \frac{\kappa}{6 \sinh \dot{d}} \left(\cosh \dot{d} \ \rho_{+} + \rho_{-} \right), \qquad (5.129)$$

in perfect agreement with the five-dimensional result (5.12). Replacing d with -d, and ρ_{\pm} by ρ_{\mp} we would get the corresponding Friedman equation for the negative tension brane, which would as well be in complete agreement with the exact five-dimensional result on the negative tension brane (5.14). The modified Friedmann equation on the brane gives precisely the exact result predicted from the five-dimensional treatment of the background. This is a non-trivial check on the validity of this close-brane four-dimensional theory as it reproduces the right result to all order in velocities with the right coupling to matter in the small distance limit and not only to leading order in \dot{d} as was the case for the low-energy effective theory. This is a very strong check that allows us to consider this theory seriously and study its prediction for perturbations where the five-dimensional geometry can not be solved exactly and therefore very little analytical progress can be made unless such an effective four-dimensional theory is used.

5.5 Perturbations Generated by Stiff Matter Source

One of the most interesting application of the close-brane theory is the study of perturbations that could arise close to a brane collision. The study of perturbations through a collision has been subject to numerous polemics however we shall be interested, for now, in the well defined question of the propagation of perturbations just before or just after such a collision. What happens at the collision itself will be beyond the scoop of this section. We motivate this study by the proposal that the Big Bang could have been generated by a brane collision. In such a model perturbations occurring at the beginning of the Universe, just after the brane collision could be at the origin of the observed large scale structure. It seems therefore essential to study the way perturbations would evolve in such a braneworld scenario. In particular in this scenario, the low-energy approximation would most certainly be violated at the beginning of the Universe, however the close-brane limit would be respected. This gives therefore a strong argument for the validity of the close-brane theory in this model.

In what follows we shall give a few examples of how perturbations propagate and point out some interesting features.

First we study in this section the perturbations generated by a stiff matter source *ie*. we consider matter to be introduced only at the perturbed level:

$$T^{\mu}_{\nu} = T^{(0)\,\mu}_{\ \nu} + \delta \,T^{\mu}_{\nu}, \tag{5.130}$$

where $T^{(0)}{}^{\mu}_{\nu}$ is the background matter contribution: $T^{(0)}{}^{\mu}_{\nu} = 0$. The background solution is that obtained from (5.127), (5.129) in the absence of matter. Furthermore the successive action of the operator \hat{O} on a perturbed tensor is straightforward. Indeed, if δZ^{μ}_{ν} is a tensor of first order in perturbations, then to that order, the action of the operator \hat{O} on this tensor is:

$$\hat{O}\,\delta Z^{\mu}_{\nu} = \left|\partial d\right|^2 \left(\begin{array}{cc} -\delta Z^0_0 & 0\\ 0 & \delta Z^i_j \end{array}\right). \tag{5.131}$$

Throughout we work only with the positive-tension brane, assuming the negative-tension brane to be empty, and drop the \pm signs. The matter contribution in the modified Einstein equation (5.100) is therefore:

$$\sqrt{\hat{O}} \, \frac{\cosh\sqrt{\hat{O}}}{\sinh\sqrt{\hat{O}}} \, \delta \tilde{T}^{\mu}_{\,\nu} = \left(\begin{array}{cc} |\partial d| \, \frac{\cos|\partial d|}{\sin|\partial d|} \, \delta \tilde{T}^{0}_{\,0} & \delta \tilde{T}^{0}_{\,j} \\ \delta \tilde{T}^{i}_{\,0} & |\partial d| \, \frac{\cosh|\partial d|}{\sinh|\partial d|} \, \delta \tilde{T}^{i}_{\,j} \end{array} \right). \tag{5.132}$$

The effect of matter at the perturbed level can therefore be tracked down exactly.

5.5.1 Tensor Perturbations

The study of tensor perturbations around a FRW background is remarkably straightforward. We consider the metric perturbation:

$$ds^{2} = -a^{2}(\tau)d\tau^{2} + a^{2}(\tau)\left(\delta_{ij} + h_{ij}\right)dx^{i}dx^{j}, \qquad (5.133)$$

where the three-dimensional tensor h_{ij} is transverse and traceless $h_i^i = 0$, $\partial_i h_j^i = 0$, and indices are raised with δ^{ij} . We assume that these gravity waves are sources by tensor matter δT_j^i at the perturbative level, with $\delta T = \delta T_i^i = 0$. Furthermore for tensor perturbations, the radion is not perturbed, $\delta d = 0$, and $|\partial d| = \dot{d}$.

For tensor perturbations, in the frame (5.133) the perturbed Ricci tensor is:

$$\delta R_{ij} = -\frac{1}{2} \bar{\Box} h_{ij} + a\dot{a} \,\dot{h}_{ij} + \dot{a}^2 \,h_{ij}, \qquad (5.134)$$

where $\overline{\Box}$ designate the Laplacian on Minkowski space: $\overline{\Box} = -\partial_{\tau}^2 + \nabla_x^2 = -a^2 \partial_t^2 - a\dot{a}\partial_t + \nabla_x^2$. Using this expression of the Ricci tensor in the modified Einstein equation (5.100) with the relation (5.132), we have:

$$\delta R_{ij} = \frac{1}{d} \Big[\delta(D_i D_j d) + \frac{\dot{d}}{L} \tanh \frac{\dot{d}}{2} a^2 h_{ij} + \frac{\kappa}{2} \dot{d} \coth \dot{d} \delta T_{ij} \Big], \qquad (5.135)$$

with $\delta(D_i D_j d) = -a^2 \frac{\dot{d}}{2} \dot{h}_{ij} - a\dot{a} \dot{d} h_{ij}$. For an empty background, the relations (5.12) or (5.129) imply the equality at small d:

$$LH_{+} = \tanh\frac{\dot{d}}{2}.\tag{5.136}$$

Putting all this together, we therefore have the equation for the tensor perturbations:

$$\hat{\boxdot} h_{ij} = \left[\bar{\Box} - \frac{d'}{d}\partial_{\tau}\right] h_{ij} = -\frac{\kappa}{d} \, \dot{d} \coth \dot{d} \, \delta T_{ij}, \qquad (5.137)$$

where $d' = a\dot{d}$ is the derivative of d with respect to the conformal time τ . The operator $\hat{\Box}$ is the same operator that would be expected from the low-energy effective theory. Indeed, the scale factor \bar{a} in the Einstein frame¹ is related to the scale factor a in the positive-tension brane frame by (Cf. section 2.6.5)

$$\bar{a}^2 = \frac{a^2}{1 - e^{-2d/L}}.$$
(5.138)

In the small d limit, $\frac{d'}{d} \simeq 2\frac{\bar{a}'}{\bar{a}}$, which is precisely the factor we would get in the operator $\hat{\boxdot}$ in the low-energy theory: $\hat{\boxdot} = \bar{\Box} - 2\frac{\bar{a}'}{\bar{a}} \partial_{\tau}$.

The main point to notice is that the effective four-dimensional Newton constant on the positive-tension brane is related to the five-dimensional one by:

$$\kappa_{4d}^{(+)} = \frac{\dot{d}}{d} \coth \dot{d} \ \kappa. \tag{5.139}$$

As it is the case in the low-energy effective theory, the coupling to matter is different for the background as it is for the perturbations - for the background, the coupling can be identified from (2.28) or (5.118) as κ/L , as opposed to (5.139). When either the branes are stabilised, and d is not treated as a dynamical variable, $d \sim d_0 = \text{const}$, or the velocity is small $\dot{d} \ll 1$ (which is the case in the low-energy limit), we recover the standard result:

$$\kappa_{4d}^{(+)} \sim \frac{\kappa}{d_0}.\tag{5.140}$$

¹Strictly speaking the notion of "Einstein frame" is meaningless beyond the low-energy approximation since there is no such a frame for which the equations of motion derive from the action of a scalar field minimally coupled to gravity. However we consider here the frame which is conformally related to the brane frame in the same way as it is in the low-energy effective theory in (2.51).

However, for arbitrary brane velocities, when the radion is not stabilised, the exact result for small d is given by (5.139). As expected, the effective Newton constant picks up a dependence on d, as it does in the low-energy theory, but more unexpected, it also contains some degree of freedom: the brane separation velocity. Whilst this is not expected to be relevant today, since one would assume the radion is stabilised in the present Universe, it would be extremely important near the brane collision. As discussed in section 5.2, \dot{d} would be approximately constant, $\dot{d} \sim v$, leading to

$$\kappa_{4d}^{(+)} \sim \frac{\coth v}{t} \,\kappa,\tag{5.141}$$

where the coefficient $\coth v$ could take any value greater than 1 depending on the matter content of the branes.

5.5.2 Scalar Perturbations

We now consider the scalar perturbations generated by the presence of a perfect fluid on the positive brane:

$$T_{\mu\nu} = (\rho + p)u_{\mu}u_{\nu} + p q_{\mu\nu}^{(+)}.$$
(5.142)

where u_{μ} is the four-velocity $u_{\mu}u^{\mu} = -1$ and p and ρ its pressure and energy density. We suppose that the background contribution of this fluid cancels (again, the background geometry is taken to be empty). For scalar perturbations, we therefore have from (5.132)

$$\sqrt{\hat{O}} \frac{\cosh\sqrt{\hat{O}}}{\sinh\sqrt{\hat{O}}} \tilde{T}^{\mu}_{\nu} = \begin{pmatrix} -|\partial d| \frac{\cos|\partial d|}{\sin|\partial d|} \left(\frac{2}{3}\delta\rho + \delta p\right) \\ \frac{|\partial d|}{3} \frac{\cosh|\partial d|}{\sinh|\partial d|} \delta\rho \end{pmatrix}.$$
 (5.143)

The presence of this fluid will generate scalar perturbations in the metric. We choose to work in comoving gauge for which the distance d between the brane is unperturbed $\delta d = 0$ *ie.* we evaluate the perturbations on hypersurfaces of constant d. The metric can be expressed in the form:

$$ds^{2} = a^{2} \Big((-1+2\Phi) d\tau^{2} + 4E_{,i} d\tau dx^{i} + (1+2\Psi) \delta_{ij} dx^{i} dx^{j} \Big).$$
(5.144)

In that gauge we then have:

$$\delta \left| \partial d \right| = \dot{d} \Phi. \tag{5.145}$$

From the expression of the Ricci scalar (5.101), we get:

$$\delta R = \frac{\kappa}{L} \left(\delta \rho - 3\delta p \right). \tag{5.146}$$

Working in terms of the conformal time (with $' \equiv \partial_{\tau}$), we have a'' = 0 for the background (since we assumed the brane to be empty and spatially flat). Deriving the expression for the perturbed Ricci tensor in (5.146), we have:

$$\Psi'' + \frac{a'}{a} \left(2k^2 E + \Phi' + 3\Psi' \right) + \frac{k^2}{3} \left(2\Psi + 2E' - \Phi \right) = a^2 \frac{\kappa}{6L} \left(\delta\rho - 3\delta p \right).$$
(5.147)

We now perturb (5.100), using for simplicity the notation

$$X(|\partial d|) = |\partial d| \tanh(|\partial d|/2)$$

$$Y(|\partial d|) = \frac{1}{|\partial d|} (\tanh(|\partial d|/2) + \tan(|\partial d|/2)).$$

The ij (with $i \neq j$) component of the Einstein equation, to first order in the perturbations, reduces to:

$$\Phi - \Psi - 2E' = \left(4\frac{a'}{a} + 2\frac{d'}{d}\right)E,$$
(5.148)

and the 0i-component to:

$$\Psi' = -\left(\frac{a'}{a} + \frac{d'}{2d}\right)\Phi.$$
(5.149)

So far these equations are equivalent to those one would have obtained in the low-energy limit. The difference comes from the 00-component of the perturbed Einstein equations:

$$\frac{d'}{d}\Phi' + \frac{a^2}{Ld} \left(2X - \dot{d}X' + \dot{d}^3Y' \right) \Phi - 2k^2\Psi - \frac{a'}{a} \left(6\Psi' + 4k^2E \right) \\ = -a\frac{\kappa}{6}\frac{d'}{d}\cot\dot{d} \left(2\delta\rho + 3\delta p \right), \qquad (5.150)$$

and from the equation of motion for d:

$$\Phi' + \left(8\frac{d'a'^2}{L}X + \frac{a}{L}(Y'\dot{d}^2 - 4X')\Phi + 3\Psi' + 2k^2E - a\frac{\kappa}{6}\left(\cot\dot{d}\left(2\delta\rho + 3\delta p\right) - 3\coth\dot{d}\delta\rho\right).$$
 (5.151)

Note that one must, at this order, treat Φ, Φ', Ψ and Ψ' as four independent variables; differentiation with respect to conformal time will miss terms arising from higher order in d, since d and d' are of different order. We must then solve the five equations (5.147-5.151) simultaneously. Using (5.148) to eliminate Φ from (5.147), we obtain

$$4k^{2}\left(\frac{d'}{d}-\frac{a'}{a}\right)E = 6\Psi''+2k^{2}\Psi+6\frac{a'}{a}\left(\Phi'+3\Psi'\right)-\frac{\kappa}{L}a^{2}\left(\delta\rho-3\delta p\right).$$
 (5.152)

We may use a combination of (5.150) and (5.151) to find an expression for Φ' in terms of $\Phi, \Psi, \delta\rho$ and δp and hence write E in terms of $\Psi, \Psi', \Psi'', \delta\rho$ and δp . This can then be used in (5.150) to obtain a complicated expression for Ψ in terms of $\Psi', \Psi'', \delta\rho$ and δp . We then only keep the leading order in d for each coefficient, resulting in the much simplified equation for the curvature perturbation on comoving hypersurfaces:

$$\hat{\boxdot} \Psi = \left[\bar{\Box} - \frac{d'}{d}\partial_{\tau}\right] \Psi = -a^2 \frac{\kappa}{6} \frac{\dot{d}}{d} \coth \dot{d} \,\delta\rho, \qquad (5.153)$$

giving rise to the same relation between the four-dimensional Newtonian constant and the five-dimensional one as in (5.139). Here again we may check that, apart from the modification of the effective Newtonian constant on the brane, the perturbations propagate in the given background exactly the same way as they would if the theory was genuinely four-dimensional. This is a very important result for the propagation of scalar perturbations if they are to generate the observed large scale structure. The five-dimensional nature of the theory does affect the background behaviour but on this given background the perturbations behave exactly the same way as they would in the four-dimensional theory.

This result is of course only true in the close-brane limit, for which the theory contains no higher than second derivatives, only powers of first derivatives. When the branes are no longer very close to each other, the theory will become higher-dimensional (in particular the theory becomes non-local in the one-brane limit). The presence of these higher-derivative corrections (not expressible as powers of first derivatives) is expected to modify the way perturbations propagate in a given background, mainly because extra Cauchy data would need to be specified, making the perturbations non adiabatic (Cf. chapter 4). However if
we consider a scenario for which the large scale structure is generated just after the brane collision, the mechanism for the production of the scalar perturbations will be very similar to the standard four-dimensional one.

5.5.3 Relation between the four- and five-dimensional Newtonian Constant

The relation (5.139) between $\kappa_{4d}^{(+)}$ and the five-dimensional constant κ is formally only valid for small distance between the branes. However if we consider the analysis of section 5.3.5, we may have some insights of what will happen if we had not stopped the expansion to leading order in d. Here, terms of the form $d\ddot{d}$ and more generally any term of the form $d^n d^{(n+1)}$ have been considered as negligible in comparison to \dot{d} and therefore only the terms of the form \dot{d}^n have been kept in the expansion. In a more general case, when the branes are not assumed to be close to each other, any term of the form $d^n d^{(n+1)}$ should be considered and would affect the relation between the four-dimensional Newtonian constant and the five-dimensional one. For moving branes, we therefore expect the relation between $\kappa_{4d}^{(+)}$ and κ to be:

$$\kappa_{4d}^{(+)} = \frac{\kappa}{L} \Omega\left(\frac{d}{L}, \dot{d}, d\ddot{d}, \cdots, d^n d^{(n+1)}\right).$$
(5.154)

The relation is therefore a functional of d: $\Omega[d(t)]$ has an infinite number of independent degree of freedom.

In the low-energy limit, or when the radion is stabilised, $d \sim d_0 = \text{const}$, the exact expression of Ω is [121]:

$$\Omega \to \Omega [d(t) = d_0] = \frac{e^{d_0/L}}{2\sinh d_0/L}.$$
 (5.155)

For close branes, another limit is now known: When $d \ll L$,

$$\Omega\left[d \ll L\right] = \frac{\dot{d}}{d} \coth \dot{d}.$$
(5.156)

But in a general case, Ω (and therefore $\kappa_{4d}^{(+)}$) is expected to be a completely dynamical degree of freedom. For the present Universe the radion is supposed to be stabilised, but in early-Universe cosmology, the effective four-dimensional Newton constant could be very

different from its present value. It might therefore be interesting to understand what the constraints on such time-variation of the Newtonian constant would be and how it would constrain the brane velocity [161], or whether such a time variation could act as a signature for the presence of extra dimensions.

5.6 Perturbations in the Presence of a Potential

We now consider perturbations in the absence of matter on the branes when a potential for the radion is present. As seen in section 2.8.1, the origin of this potential is not completely understood but could be generated by the presence of bulk scalar fields in a mechanism similar to the one proposed by Goldberger and Wise. For the purpose of this study however, we shall not worry about its origin but introduce such a potential by hand directly in the effective theory. This is similar to what has been done for the low-energy effective theory (giving rise to the modified Einstein equation on the brane (2.85) and the Klein-Gordon equation for the radion (2.86)). The same procedure can now be applied to the theory in the close-brane limit. For simplicity we will consider the branes to be empty. And work in the positive tension brane frame, unless specified otherwise. The superscript (+) will be omitted to lighten the notation.

5.6.1 Introduction of a Potential to Close-brane Effective Theory

We consider the addition of a potential in the close-brane theory by modifying the effective stress-energy tensor for the radion in the Einstein equation (5.104):

$$T_{\mu\nu}^{\text{eff}} = \frac{1}{d} \Big[D_{\mu} D_{\nu} d + \frac{2}{L} g \,\partial_{\mu} d\partial_{\nu} d - d \,\bar{E}_{\mu\nu} - \left(\Box \,d + \frac{z^2 g}{L} + \frac{3f}{L} - d \,\bar{E} + V(d) \right) q_{\mu\nu} \Big], \,(5.157)$$

where the potential V is related to the potential U introduced in the low-energy effective theory by $V(d) = 2d^2/L U(\phi(d))$ in the close-brane limit. For this stress-energy to remain conserved after the addition of the potential, the equation of motion of the scalar field (5.105) has to be modified as well:

$$\Box d = -\frac{1}{L} \left(2z^2 g + 4f \right) + d\bar{E} + W, \qquad (5.158)$$

where the correction term W is a functional of the potential V(d) to be determined. The modified Klein Gordon equation (5.158) should be consistent with the conservation of the modified stress-energy tensor in (5.157). The divergence of $T_{\mu\nu}^{\text{eff}}$ can be calculated using the constraint (5.108) for $\bar{E}_{\mu\nu}$:

$$dD_{\mu}T_{\nu}^{\mu \text{ eff}} = \partial_{\nu}d\left[-V'(d) + \frac{3W+4V}{2d} + \frac{2g}{L}\left(3W+2V\right) + \frac{12}{L}z^{2}\left(f'+z^{2}g'\right)\left(V+W\right)\right] + \frac{1}{L^{2}}\mathcal{O}(d/L).$$
(5.159)

To lowest order in d/L, we therefore need to modify the equation for the d with a term W:

$$W = \frac{2}{3} \left(dV'(d) - 2V(d) \right), \tag{5.160}$$

where all covariant derivatives and index raising is taken with respect to $q_{\mu\nu}^{(+)}$. This is precisely the same term which had to be introduced in the Klein Gordon equation (2.86) in the effective low-energy theory, in the presence of a potential. Indeed, using the relation between the potential U in (2.85) and the potential V in (5.157) for close branes, we have $V(d) = 2d^2/L U(\phi(d))$ so $W = \frac{2}{3} (dV'(d) - 2V(d)) = \frac{4}{3} \frac{d^3}{L} U_{,d}$ which is the small d limit of $\frac{L^2}{6\Psi^2} (1 - \Psi^2)^3 U_{,d}$ in (2.86). This procedure is completely consistent with the low-energy theory and will give the same result to leading order in (∂d) . In the absence of matter, the scalar curvature does not vanish anymore but couples to the potential as

$$R = 2V'(d). (5.161)$$

We have shown that it is therefore possible to modify the close-brane theory by introducing a potential in such a way that the conservation of energy and thus the Bianchi identity remains unaffected. When no matter is present, it is therefore possible to extend the close-brane theory in order to study the possibility of a potential for the radion. So far no constraint has been imposed on the potential V (as long as its order of magnitude is comparable with the other terms of the theory), however if V satisfies some slow-roll or fast-roll conditions as explained in chapters 3 and 4, production of a scale invariant spectrum might be possible. The inflaton could then have a natural physical interpretation as representing the distance between the branes. The five-dimensional nature of the theory represents the key feature for the interpretation of the scalar field, it is therefore essential to understand the higher-dimensional effects that such a scenario would have on the production of a scale invariant spectrum. We suggest to study these effects (if any) in the close-brane regime, where the effective theory on the brane seems genuinely different from a standard theory of gravity in four-dimension.

5.6.2 Production of a Scale-invariant Power Spectrum in the Close-brane Limit

Background behaviour

In order to study the perturbations in the close-brane theory with potential, we first consider the background behaviour. For this we assume cosmological symmetry and consider the positive tension brane to be spatially flat. Working in the conformally flat spacetime

$$\mathrm{d}s^2 = a^2 \eta_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu, \qquad (5.162)$$

in terms of the conformal time τ , the Ricci scalar (5.161) is

$$R = 6\frac{a''}{a^3} = 2V'(d), \tag{5.163}$$

and the equation of motion for the radion (that we will consider to play the rôle of the radion) is given by (5.158):

$$\ddot{d} = -3H\dot{d} + \frac{1}{L}\left(2\dot{d}^2g(z) + 4f(z)\right) - \frac{2}{3}\left(dV'(d) - 2V(d)\right),\tag{5.164}$$

with $z = \dot{d} = \frac{d'}{a}$. Using (5.163) and (5.164), the relation between the Hubble parameter and the brane velocity in the presence of a potential is:

$$H^{2} = \frac{\dot{d}}{dL} \left(\tanh \frac{\dot{d}}{2} - L H \right) + \frac{1}{6d} V.$$
(5.165)

In the absence of a potential this was leading to the relation between H and \dot{d} : $LH = \tanh \frac{\dot{d}}{2}$, but if the potential V is of the same order of magnitude than H or \dot{d} (in particular

if V does not diverge at the collision, when $d \to 0$), this leads instead to

$$LH = \tanh\frac{\dot{d}}{2} + \frac{L}{6\dot{d}}V.$$
(5.166)

As expected, the presence of the potential modifies the background behaviour. For appropriate conditions on V, the Hubble parameter might evolve in such a way that the production of a scale invariant spectrum in a contracting or expanding universe might be possible. For a given potential, the background behaviour will be different than a standard four-dimensional one. However we do not concentrate on the specific constraints the potential should have and we assume that it possible to tune the potential such that the Hubble parameter satisfies the required background evolution. Instead, we consider the propagation of perturbations on that given background and check whether the perturbations evolution for one given background is any different from the one in standard four-dimensional gravity.

Scalar Perturbation

In order to study scalar perturbations, we work in comoving gauge for which $\delta d = 0$:

$$ds^{2} = a(\tau)^{2} \left(\eta_{\mu\nu} + h_{\mu\nu}\right) dx^{\mu} dx^{\nu}, \qquad (5.167)$$

with $h_{00} = 2\Phi$, $h_{0i} = 2E_{,i}$ and $h_{ij} = 2\Psi\delta_{ij}$. In that gauge, the curvature perturbation on *comoving* hypersurfaces is simply $\zeta = \Psi$. Following the same procedure as in section 5.5.2, we find the remarkably simple evolution equation for the curvature perturbation:

$$\zeta'' + \frac{d'}{d}\zeta' + k^2\zeta = 0, (5.168)$$

where derivatives are performed with respect to the conformal time. The derivation of this result does not need to be shown in detail as no main features differ from the derivation of the scalar perturbations in section 5.5.2. However it is worth pointing out that the derivation and the final result does not depend on the exact expression of the functions f(z) and g(z).

This is a remarkable result which is surprisingly similar to what would be obtained in standard four-dimensional gravity with a scalar field. We have already pointed out that in the effective low-energy theory, the scale factor \bar{a} in the Einstein frame is related to the scale factor in the positive tension brane frame by: $\bar{a} = \sqrt{\frac{2d}{L}} a$, so that $\frac{d'}{d} \approx 2\frac{\bar{a}'}{\bar{a}}$, which in the small d limit corresponds exactly with what we have from a four-dimensional theory:

$$\zeta'' + 2\frac{\bar{a}'}{\bar{a}}\,\zeta' + k^2\zeta = 0. \tag{5.169}$$

This seems to suggest that the curvature propagation will not be affected by the fivedimensional nature of the theory in the close-brane limit.

Consequences for brane inflation

In the context of inflation, if the potential V is such that the Hubble parameter satisfies slow-roll conditions, or more specifically if the potential is such that $\frac{\bar{a}'}{\bar{a}} \sim \tau^{-1}$, ie. $\bar{a} \sim -(\bar{H}\tau)^{-1}$, the curvature perturbation will evolve following the same behaviour as in standard inflation and will therefore generate a scale invariant power spectrum which could be responsible for the present large scale structure. On the down side, using the close-brane theory, it seems difficult to track any characteristic signature of such a brane inflation model in the large scale structure. One of the reason why the resulting perturbation evolution seems to be so similar to a normal evolution comes from the fact that the close-brane theory differs by the presence of higher-order derivatives terms which can all be expressed as a function of $|\partial d|$. As it is always possible to work in a gauge for which $\left|\partial d\right|$ is unperturbed, only the background will be strongly affected by the presence of this functions. At the perturbed level, on the other hand, the differences from standard fourdimensional gravity can always be reinterpreted as background effects. So by modifying the background behaviour, the evolution of the perturbations will appear unaffected. This is however only valid to leading order in d/L where terms of the form $d \partial_{\mu} \partial_{\nu} d$ are considered to be negligible in comparison to $\partial_{\mu} d \partial_{\nu} d$. When the theory is genuinely of higher order in derivative *ie.* including terms which can not be expressed in terms of first or second order derivative, this result might not hold any longer. However we have seen in chapter 4, that the presence of quadratic terms in the Weyl tensor, (which are genuinely higher order in derivatives) did not seem to affect brane inflation, but could on the other hand affect the production of a scale invariant spectrum in a fast-roll potential.

Consequences for a fast-roll potential

It is worth pointing out that the result in (5.169) corresponds exactly with the result used in a number of work [90, 91] and in particular by P. Creminelli et. al. in [156]. Their argument is therefore completely valid. In particular they show that for an expanding universe with slow-roll constraints on the background behaviour, there will be production of a scale invariant spectrum. We have shown here that the five-dimensional nature of the theory does not affect the result provided the large scale structure is produced in a regime where the branes are very close to each other. On the other hand, in the case of a fast-roll background behaviour, which necessities a contracting universe, no scale invariant spectrum seems to be observable. The reason why there seems to be production of a scale-invariant spectrum as shown in the previous chapter, section 4.3.3, seems to be a gauge artifact. In section 4.3.3, we have indeed worked in Newtonian gauge, for which the variable u in eq.(4.57) seems to acquire a scale invariant growing mode. However, as shown in [156], this is due to the fact that the wrong variable is used to study the power spectrum. Working in terms of the curvature perturbation in comoving gauge, the propagation evolution satisfies (5.169) which generates a scale invariant power spectrum only if $\bar{a} \sim \tau^{-1}$ and therefore does not produce any scale invariant power spectrum in the fast-roll contracting case for which $\bar{a} \sim \tau^{2\epsilon_F/(1-2\epsilon_F)}$, where $\epsilon_F \ll 1$. Their argument was relying on the fact that a four-dimensional theory was valid to study the evolution of the curvature perturbation, which could be contestable. But we have shown here that this is indeed the case, if the branes are close enough to each other. We can therefore argue that their result would be reliable in that case and no scale invariant power spectrum seems to be produced in such a fast-roll scenario.

This conclusion can appear to be in contradiction with the one of [154], where they con-

clude that a scale invariant can indeed be produced in this scenario using five-dimensional arguments. We can however point out that their result was relying strongly on the validity of the moduli space approximation, or low-energy effective theory, which we have shown to be valid only to second order in velocity. Their result, however predicts the production of a scale invariant power spectrum with coefficients to fourth order in velocities, which appears to be much beyond the regime of validity of the moduli-space approximation, and therefore can not be entirely trusted.

To finish this chapter on close-brane theory, we shall end up by giving in next the section an extra argument confirming that the terms included in chapter 4 are indeed generic to braneworld cosmology and can be interpreted as part of the first Kaluza Klein modes.

5.7 First Kaluza Klein Mode in the Close-brane Limit

We now work with the same philosophy as [112] in order to find the next to leading order term in the low-energy effective theory of section 2.6, and see in particular if, in the closebrane limit, it is possible to work out the covariant expression of the first KK mode as derived in the quasi-static limit in section 2.7. For this we consider matter on the positive tension brane only.

In order to find the contribution of the Weyl tensor E^{μ}_{ν} for the zero mode (given by the low-energy effective theory in (2.46)) as well as the first KK mode, we may use the same technique as before which is to expand the extrinsic curvature on the negative brane as a Taylor series around the positive brane

$$K^{(-)\,\mu}_{\ \nu} - K^{(+)\,\mu}_{\ \nu} = \frac{\kappa}{2} \tilde{T}^{\mu}_{\nu} = \partial_y K^{(+)\,\mu}_{\ \nu} + \frac{1}{2} \partial_y^2 K^{(+)\,\mu}_{\ \nu} + \frac{1}{6} \partial_y^3 K^{(+)\,\mu}_{\ \nu} + \cdots, \qquad (5.170)$$

where \tilde{T}^{μ}_{ν} is associated to the stress-energy tensor of matter on the positive tension brane and the first equality is a consequence of the Israël matching conditions (5.90) and (5.91). For simplicity, we will work in what follows in a low-energy limit, where the quadratic (and cubic) terms in the stress-energy tensor can be neglected as well as the coupling of matter with the radion. Furthermore we assume that we can work in a gauge (5.27) where the branes are static and for which A, the yy-component of the metric, can be considered to be independent of y. In that case, using the evolution equations for the extrinsic curvature (2.5) and the Weyl tensor (2.33) are

$$\partial_y K^{\mu}_{\nu} = -d E^{\mu}_{\nu} - D^{\mu} D_{\nu} d - d K^{\mu}_{\alpha} K^{\alpha}_{\nu} + \frac{d}{L^2} \delta^{\mu}_{\nu}$$
(5.171)

$$\partial_{y}E^{\mu}_{\nu} = 2dK^{\alpha}_{\nu}E^{\mu}_{\alpha} - d\frac{3}{2}KE^{\mu}_{\nu} - \frac{d}{2}K^{\alpha}_{\beta}E^{\beta}_{\alpha}\delta^{\mu}_{\nu} + d\hat{U}K^{\mu}_{\nu} + d(K^{3})^{\mu}_{\nu} \qquad (5.172)$$
$$-\partial^{\alpha}d\left[D^{\mu}K_{\alpha\nu} + D_{\nu}K^{\mu}_{\alpha} - 2D_{\alpha}K^{\mu}_{\nu}\right].$$

where again $(K^3)^{\mu}_{\nu}$ are some cubic terms in the traceless part of the extrinsic curvature and the operator \hat{U} is such that for any tensor Z^{μ}_{ν} ,

$$\hat{U}Z^{\mu}_{\nu} = C^{\mu}_{\ \alpha\nu\beta}Z^{\alpha\beta} - \frac{1}{2} \left[D_{\alpha}D^{\mu}Z^{\alpha}_{\ \nu} + D_{\alpha}D_{\nu}Z^{\alpha\mu} - 2\Box Z^{\mu}_{\ \nu} \right].$$
(5.173)

We can now differentiate the extrinsic curvature to third order, using the relations (5.171) and (5.172) and neglecting any term of the form $\partial d\tilde{T}$ or any higher order in \tilde{T} . We find the following relations evaluated on the positive brane:

$$\partial_y K^{(+)\,\mu}_{\ \nu} = -d \, E^{\mu}_{\nu}(0) - D^{\mu} D_{\nu} d + \frac{2d}{L} \kappa \tilde{T}^{\mu}_{\nu} \tag{5.174}$$

$$\partial_y^2 K^{(+)\,\mu}_{\ \nu} = d^2 \chi^{\mu}_{\ \nu}(0) - 4 \frac{d}{L} D^{\mu} D_{\nu} d - \frac{2}{L} \partial^{\mu} d\partial_{\nu} d + \frac{1}{L} (\partial d)^2 \delta^{\mu}_{\ \nu} - \frac{6d^2}{L} E^{\mu}_{\ \nu}(0) \quad (5.175)$$

$$\partial_y^3 K^{(+)\,\mu}_{\ \nu} = \frac{d^3}{L^3} B^{\mu}_{\nu} + \frac{d^3}{L^2} \,\hat{h} E^{\mu}_{\nu}(0) + \frac{d^3}{L^3} \,\hat{g} \left(E^2(0)\right)^{\mu}_{\nu}, \tag{5.176}$$

with $\chi^{\mu}_{\nu}(0) = -\frac{\kappa}{2} \hat{U} \tilde{T}^{\mu}_{\nu} - 2\frac{\kappa}{L^2} \tilde{T}^{\mu}_{\nu}$ and $\hat{h} = L^2 \hat{U} - 28$. By $(E^2(0))^{\mu}_{\nu}$ we designate quadratic term in the Weyl tensor on which the operator \hat{g} acts. The exact action of \hat{g} is not necessary as it is negligible at low-energy. The tensor B^{μ}_{ν} can be expressed in terms of the radion and the extrinsic curvature and does not depend explicitly on E^{μ}_{ν} , for this study we do not need to compute its exact expression.

Using these expressions in the Taylor expansion (5.170) to third order in d, we therefore get the equation for E^{μ}_{ν} :

$$d(1+3\frac{d}{L}-\frac{d^{2}}{6L^{2}}\hat{h})E_{\nu}^{\mu}(0) = \frac{\kappa}{2}\left(1+2\frac{d}{L}\right)\tilde{T}_{\nu}^{\mu}+\frac{d^{2}}{2}\chi_{\nu}^{\mu}(0) \qquad (5.177)$$
$$-D^{\mu}D_{\nu}d-2\frac{d}{L}D^{\mu}D_{\nu}d-\frac{1}{L}\left(\partial^{\mu}d\partial_{\nu}d-\frac{1}{2}(\partial d)^{2}\delta_{\nu}^{\mu}\right),$$

where we have omitted terms coming from higher derivatives of the extrinsic curvature and terms that are negligible at low-energy. If we consider the Weyl tensor to be of the form $E^{\mu}_{\nu} = E^{\text{eff}\,\mu}_{\ \nu} + E^{KK\,\mu}_{\ \nu}$ where $E^{\text{eff}\,\mu}_{\ \nu}$ is the Weyl tensor derived in the low-energy effective theory (2.46), which corresponds to the zero mode in section 2.7. The KK modes should thus be encoded in $E^{KK\,\mu}_{\ \nu}$. Substituting this expression back into the equation (5.177) for E^{μ}_{ν} we get:

$$E^{KK\,\mu}_{\ \nu} = -\frac{\kappa}{6} \, d\,\hat{U}\tilde{T}^{\mu}_{\nu},\tag{5.178}$$

which should correspond to the first KK mode at low-energy, and to lowest order in d.

We may now compare this result with what is found in the quasi-static limit. We have shown in section 4.2.1, that in the quasi-static limit, the first KK mode was correctly reproduced by the term $\frac{1}{1-\Psi^2}E_{\mu\nu}^{\text{corr}}$ with

$$E_{\mu\nu}^{\rm corr} = \alpha L^2 A_{\mu\nu}, \tag{5.179}$$

and $\alpha = \frac{2d^3}{3L^3} + \mathcal{O}(\frac{d^4}{L^4})$. Where the tensor $A_{\mu\nu}$ was the one derived from the action $\int dx^4 \sqrt{-q} C^2$, given in (4.4) or (4.6). When the evolution of the radion is negligible, we indeed have, to linear order, the standard Einstein equation on the brane $R_{\mu\nu} = \frac{\kappa}{L} \frac{1}{1-\Psi^2} \left(T_{\mu\nu} - \frac{1}{2}Tq_{\mu\nu}\right)$, so we have $\tilde{T}_{\mu\nu} \approx \frac{2d}{\kappa} \left(R_{\mu\nu} - \frac{1}{6}Rq_{\mu\nu}\right)$. Substituting this into the expression (5.178) for the first KK mode, we therefore have:

$$E^{KK\mu}_{\nu} = -\frac{d^2}{3} d\hat{U} \left(R^{\mu}_{\nu} - \frac{1}{6} R \delta^{\mu}_{\nu} \right)$$

= $-\frac{d^2}{3} C^{\mu}_{\ \alpha\nu\beta} R^{\alpha\beta} + \frac{d^2}{6} \left[D_{\alpha} D^{\mu} R^{\alpha}_{\nu} + D_{\alpha} D_{\nu} R^{\alpha\mu} - 2 \Box R^{\mu}_{\nu} \right]$ (5.180)
 $-\frac{d^2}{18} \left[D^{\mu} D_{\nu} R - 2 \Box R \delta^{\mu}_{\nu} \right].$

Decomposing the Weyl tensor $C^{\mu}_{\ \alpha\nu\beta}$ in terms of the Riemann tensor, the Ricci tensor of the scalar curvature, and using the fact that $D_{\alpha}D^{\mu}R^{\alpha}_{\ \nu} = \frac{1}{2}D^{\mu}D_{\nu}R + R^{\mu}_{\ \alpha}R^{\alpha}_{\ \nu} - R^{\mu}_{\ \alpha\nu\beta}R^{\alpha\beta}$, we obtain:

$$E^{KK\mu}_{\ \nu} = -\frac{d^2}{3} \left[2R^{\mu}_{\ \alpha\nu\beta}R^{\alpha\beta} + \Box R^{\mu}_{\ \nu} - \frac{1}{3}D^{\mu}D_{\nu}R - \frac{1}{6}\Box R\delta^{\mu}_{\ \nu} + \frac{1}{6}R^2\delta^{\mu}_{\ \nu} - \frac{2}{3}RR^{\mu}_{\ \nu} - \frac{1}{2}R_{\alpha\beta}R^{\alpha\beta}\delta^{\mu}_{\ \nu} \right],$$
(5.181)

which corresponds precisely to the tensor $A_{\mu\nu}$ we took as an ansatz for the first KK mode in section 4.2.1:

$$E_{\mu\nu}^{KK} = \frac{d^2}{3} A_{\mu\nu} = \frac{1}{1 - \Psi^2} E_{\mu\nu}^{\text{corr}}.$$
 (5.182)

This is therefore a strong argument confirming that the ansatz we took in chapter 4 to model part of the first KK mode is consistent with the five-dimensional covariant derivation of this mode. In particular in the section 4.3.3, we worked in a low-energy regime and studied the effect of this mode in the duality between fast-roll and slow-roll. For that low-energy study, this derivation confirms the validity of our ansatz and therefore the conclusion that were drawn on that section. In a more general case, however the term $E^{KK\mu}_{\ \nu} = -\frac{\kappa}{6} d\hat{U}\tilde{T}^{\mu}_{\nu}$ should be considered as representing part of the first KK mode in the close-brane limit.

5.8 Discussion

In the first part of this chapter we pointed out the discrepancy between predictions from the effective four-dimensional low-energy theory and the exact five-dimensional results. The difference has been established in the regime where the branes were close and all through this paper we worked to leading order in d/L.

In order to go beyond the low-energy effective theory, we established a formalism to find the electric part $E_{\mu\nu}$ of the five-dimensional Weyl tensor on both branes. This tensor represents the only quantity which is unknown from a four-dimensional point of view as it encodes information about the bulk geometry. Finding its expression on the brane is therefore the key element in order to study the geometry of a brane within an extra dimension.

Using this formalism in the small distance limit, we found an exact expression for the Weyl tensor on each brane, valid at leading order in d/L but at all orders in velocities, or for any energy scale. We were therefore able to modify the regime of validity of the effective theory from a low-energy regime valid independently of the branes distance to a regime

valid at all energy scales for close branes. This regime of validity is relevant for cosmology since it represents one of the main focus of present braneworld models. Understanding the behaviour of branes just before or just after a collision is indeed crucial if our Universe emerged from a brane collision. If the Big Bang were initiated by such a collision, it is consistent to assume that the large scale structure was produced in a regime where both branes were close. Even if this regime is not valid after a while, effects that originated during this period are unlikely to be eliminated once the branes are far apart and the fifth dimension has opened up, leaving the possibility of exciting the Kaluza Klein modes.

Keeping this objective in mind, we established a theory that allows us to study such scenarios and have checked the consistency of its predictions for homogeneous and isotropic geometries. We argue that this theory is remarkably straight-forward to use since it includes only four-dimensional quantities and is effectively second order in derivatives, the only higher-order corrections coming as powers of first derivatives. This feature will facilitate any comparison with other four-dimensional theories. It is worth pointing out that it is different from other higher-derivative theories [150, 162] in that it is not derived using the assumption that it can be derived from an action, and it is different from a theory relying on purely string effects [81, 163, 164]. Already for the background, an interesting result can be pointed out: when the branes are empty the Hubble parameter on each brane is bounded by $L^2H^2 < 1$, which could not have been derived from the low-energy effective theory, although this does not hold anymore when some matter is introduced on the branes or if a potential is added for the radion. Another interesting result of the theory is that the expansion seems to break down when the velocities are of order $|\partial d| \sim \pi$. However when any physical quantity is calculated (for the background as much as for perturbations), the terms of the form $\sin |\partial d|^{-1}$ do not contribute and the result is limited by any value of $|\partial d|$.

We can then use this close-brane effective theory in order to understand the way matter and the radion couples to gravity at the perturbed level. In order to do so, we considered a scenario in the stiff source approximation for which the background is supposed to be unaffected by the presence of matter and considered the production of curvature and tensor perturbation sourced by the presence of matter fields on the brane. Although the fivedimensional nature of the theory does affect the background behaviour, we have shown that for a given background the perturbations propagate the same way as they would in a standard four-dimensional theory. This is only true in the limit of small brane separation and is not expected to be valid outside this regime. However, since the large scale structure of the Universe might have been produced in a period for which the branes could have been close together (for instance just after a brane collision initiating the Big Bang), this regime is of special interest. The fact that the perturbations behave the same way, for a given background, as they would in a four-dimensional theory is a remarkable result for the production of the large scale structure which could be almost unaffected by the presence of the fifth dimension. This is true for an inflationary scenario. On the other hand in a fast-roll scenario, this confirms the fact that no scale invariant power spectrum seems to be produced. This is however only valid if we consider the perturbations to be generated in a regime where the branes are close enough.

The relation between the four-dimensional Newtonian constant and the five-dimensional one is however affected by the expansion of the fifth-dimension. As we have shown in the Introduction of this thesis and in chapter 2, as well as in [50, 121], the four-dimensional Newtonian constant is dependent on the distance between the branes, giving a possible explanation of the hierarchy problem. In this chapter, we have shown that the fourdimensional Newtonian constant also has some dependence on the brane velocity which we computed exactly in the small-distance limit. This might provide us with an observational signature for the presence of extra dimensions. Outside the small *d* regime, we expect the four-dimensional Newtonian constant to depend on the five-dimensional one not only through the brane separation velocity \dot{d} but also on higher derivatives of the distance between the branes $d^{(n)}$, making the requirement for moduli stabilisation even more fundamental for any realistic cosmological setup within braneworld cosmology.

Chapter 6

Conclusions



Ah Puch, the Underworld (Maya)

We end by summarising in this chapter the main points of this thesis. As we shall see, there are several criticisms that can be made either of the formalism developed or of the validity of the results. We shall therefore emphasise possible extensions of this work, and show how some potential problems might be cured.

6.1 Objective

The notion of low-energy effective field theories has become a familiar concept in particle physics. In these theories, the influence of higher energy degrees of freedom (or equivalently higher dimensional ones) is described by the addition of higher derivative or mass terms (irrelevant operators) in the effective action. These can be obtained by explicitly integrating out the high energy modes. This approach is valid as long as the energy scales of interest are much less than the energy scales of the modes which have been integrated out (*ie.* when these modes are very massive).

In this thesis, the approach is philosophically distinct: We have attempted to model the five-dimensional effects in four-dimensional terms, in regimes where the energy scales of interest are comparable to the five-dimensional energy scales. In these regimes, the traditional four-dimensional low-energy effective field theory approach will thus not be valid. By making various simplifying assumptions in distinct cases, we have nevertheless been able to model the five-dimensional effects with a four-dimensional theory, although this often may not be describable in terms of an action.

6.2 What did we learn ?

Although not every step of this thesis is fundamentally new, the general analysis has led to several results or concepts which are worth emphasising one last time:

- Braneworld cosmology is well described by an effective theory at low energies, but is genuinely different at high energies. The modified Einstein equation on an orbifold brane presents two characteristic terms. First gravity couples quadratically to the stress-energy tensor of matter on the brane. Secondly, Kaluza Klein modes, representing excitations from the bulk degrees of freedom, are present on the brane. These two elements (which disappear at low energies), reflect the genuinely five-dimensional nature of the theory, and hence the necessity of going beyond the low-energy description.
- The low-energy effective theory breaks down at large-velocities, even for the background where it could be believed to give an accurate result to all orders. This discrepancy is indeed independent of the Kaluza Klein modes observed at the perturbed level in the quasi-static limit and of the quadratic terms in the stress-energy.

It reflects the fact that the integration constant (arising from integration of the Einstein equation), is related to the radion dynamics (or in a more general system, to the moduli dynamics). The relation predicted by the low-energy effective theory is shown to be only valid at low velocities and to break down at second order.

To go beyond the low-energy limit, the zero mode can be correctly modelled using an *adiabatic argument* together with a *conformal assumption*. Using this assumption, we explore the possibility for the branes to be conformal to each other. In such a case a homogeneous tensor which vanishes for the background can be separated out from the Weyl tensor. Since scalar perturbations of a conserved fluid follow their background behaviour in an adiabatic regime, this homogeneous part of the Weyl tensor can be neglected as a first approximation.

In this model, Kaluza Klein modes, which break the conformal assumption, can be introduced as corrections to the homogeneous part of the Weyl tensor. By considering different corrections on each brane, the conformal assumption is hence relaxed. The treatment allows the study of Kaluza Klein type of modification to the brane geometry.

In the quasi-static limit, the first Kaluza Klein mode can be modelled by a fourdimensional local tensor. This term can be derived from an action containing quadratic terms in the four-dimensional Weyl tensor. The first Kaluza Klein mode can be modelled by the presence of this new term which is introduced with different coefficients on each brane. This model gives an exact result only in the quasi-static limit, and one can argue that in a general setup, this result should not be valid. However, it is very unlikely that its contribution could be cancelled out in generic situations. Although in general situations, the term we have considered might not be the more significant one, it should still be present in a wide range of scenarios. This term might hence be present and have an affect beyond the quasi-static limit.

In the close-brane regime, a four-dimensional effective theory can be derived. By

successive integration of the bulk equations of motion, the induced Einstein equation of the brane can be expressed covariantly and in a closed form. The resulting theory contains terms of higher orders in derivatives although they can actually be expressed as a power of first derivative terms. The Cauchy data to be specified in the initial conditions of the theory is hence the same as for a genuinely second order theory, there is no need for new information to be specified. Although it appears effectively four-dimensional, this theory precisely models five-dimensional effects of the theory and gives precisely the exact result for the background in the close-brane regime. It can as well be seen as a generalisation of the well-known low-energy fourdimensional effective theory in the close-brane limit and is hence an extension of this theory beyond the low-energy approximation.

Using these results, and the different modelling theories, **signatures in the propagation** of perturbations are shown to be very weak. Indeed, when the inflaton scalar field lives on the brane, corrections to standard four-dimensional chaotic inflation come at second order in the slow-roll parameters. The main differences arise from the background behaviour, which can always be remodelled by a change of variables. Such background differences lead to a possible reinterpretation of the potential. The constraints to apply on the potential are hence different, and in particular a scenario for steep inflation, for which the potential is not flat, is possible.

Furthermore, perturbations produced in a regime where the branes are close enough follow a general four-dimensional behaviour, although the underlying theory is genuinely different from four-dimensional scenarios. We can hence conclude that the four-dimensional treatment is usually acceptable in an adiabatic situation and gives accurate predictions.

The presence of a shear tensor can however be pointed out at high energies when the inflaton lives on the brane. This anisotropic term is usually absent in standard chaotic inflation and hence is interesting to study in braneworld inflation. Unfortunately, it does not provide any observational signatures which could characterise braneworld

scenarios.

On the other hand, higher-order derivative corrections modify the way perturbations propagate. When correction terms that could model Kaluza Klein corrections are taken into account, the behaviour of perturbations is genuinely different. In particular, perturbations propagate with a speed of sound slightly different from the speed of light, which is not necessarily the same for tensor and scalar perturbations. Another interesting difference arises in the estimate of the non-gaussianity. Although it remains small in our analysis, it does not appear to be damped by the slow-roll parameters. It hence opens up the possibility for the non-gaussianity to be relatively large in a general braneworld scenario.

In the case of this study, the corrections are considered to be small and do not affect the general features of inflation. The differences are not important enough to create an observable distinction to inflation. However, for perturbations evolving in a fast-roll potential, the production of a scale invariant power spectrum is in general affected and the slow-roll/fast-roll duality is hence broken.

A general bulk brane has an induced Einstein equation which presents different remarkable elements, not present on an orbifold brane. The presence of these terms is related to the degrees of freedom present on the brane and not fixed by the \mathbb{Z}_{2} reflection symmetry. At high energies, the coupling to matter is very different from what is usually expected from four-dimensional theories. Their study is of special interest, as their behaviour is closer to usual D-branes (putting aside the charges and form fields), and hence their study can lead to interesting features for these branes as well. When an adiabatic regime is considered, an effective four-dimensional theory can be derived on such branes, provided no matter fields are introduced. The way to derive this theory is very similar to what has been done for orbifold branes and general Kaluza Klein type of corrections could be considered as well.

6.3 Criticisms and Going forward

Although we believe the previous development to be formally correct, different criticisms can be made. We shall summarise them in what follows and see what alternatives can be proposed to extend the study. Such an extension could give rise to more interesting results. The different points one might criticise are the followings:

The treatment of the Weyl tensor. In chapters 3 and 4, the treatment of the Weyl tensor has been imposed by hand at each steps, instead of being inferred by solving the five-dimensional equations of motion like the more coherent procedure which is followed in 5. The treatment of chapters 3 and 4 does go beyond the usual assumptions, where the Weyl tensor is often simply neglected [129, 139, 144] or treated as radiation [165], but despite this fact, the way the homogeneous part of the Weyl tensor is treated is still far from an exact analysis. Part of its behaviour, is only guessed in chapter 4 and an ansatz is put by hand. This extends the treatment of [162], for instance, which assumed the presence of some generic terms in the action, without constraining these terms with the known background behaviour, nor the quasi-static limit, or any other well defined and computable regime.

The treatment of chapter 5, on the other hand presents a better way to attack the problem, in a more five-dimensional way. In particular, the last section shows how the ansatz of chapter 4 can be incorporated in a more general calculation. We believe this is a better way to treat the problem, and further works in this direction should bring better insight on the behaviour of the Weyl tensor.

The one-brane limit has been considered in chapters 3 and 4. In this limit, the negative tension brane decouples from the positive one and the study could hence be focused on the quadratic terms in the stress-energy tensor on the brane or on Kaluza Klein corrections, without needing to include the backreaction of the negative brane. But in the one-brane limit of the RS model, the notion of a discrete Kaluza Klein tower has to be replaced by a Kaluza Klein continuum. There is hence no real sense in

considering the first Kaluza Klein mode and its effect in that model. However this does not mean that corrections are not present, (Kaluza Klein effects are expected to be more important in the one-brane limit). In particular if the correction term we have considered affects the brane geometry in some limit, it is unlikely that other terms present in the Kaluza Klein continuum will cancel this effect.

In chapter 4, the correction term modelling the first Kaluza Klein mode was assumed to be small, and hence no real conclusive remarks could be clearly made.

This assumption was made because the correction generates a term of fourth order in derivatives. It is hence difficult to understand how initial conditions should be imposed and to solve the equations exactly. We may stress that this problem does not exist in other higher-derivative theories such as in [81] or in theories including Gauss-Bonnet terms [85], since the higher-derivative terms are expressible as powers of first or second order derivatives.

This problem could be solved if the equations were solved numerically and if the usual Bunch-Davis vacuum was still considered as the initial condition. Assuming this vacuum was still valid at higher order in derivatives, initial conditions could hence be imposed after differentiation of the Bunch-Davis vacuum. By using such a method, it would be possible to go beyond the small correction assumption, and hence consider the entire effect of this term. This should lead to much more interesting and conclusive remarks.

Only a first order expansion in brane separation has been considered in the closebrane theory. Although the first order expansion has been remarkably simple and straight-forward to work with, it has not led to any features that could not be obtained from a four-dimensional scenario. However, if we go beyond the first order, some genuinely higher-order derivative terms will appear and affect the behaviour of perturbations in a non-trivial way. Studying the next to leading order terms in the close-brane effective theory would hence be a natural extension to consider.

- No mechanism has been suggested to explain the presence of a potential. Such a potential is indeed needed either for inflation or to generate a scale-invariant power spectrum, but has not been derived from any theory. In chapters 3 and 4, inflation was considered on the brane where an inflaton scalar field was supposed to live. In that case, the problem is the same as for usual four-dimensional inflation: The origin of the inflaton scalar field is unknown and we only assume the existence of a potential satisfying some slow-roll constraints. In chapter 5, the radion plays the rôle of the inflaton, but the potential is introduced by hand in the effective theory. In order to take this model seriously, one should suggest a mechanism for the production of such a potential, relying on higher-dimensional features, such as the Goldberger-Wise mechanism [122], or on properties of the underlying theory such as the one considered for brane inflation, as explained in the introduction.
- We have focused our study on unrealistic models. The Randall Sundrum model is indeed nothing else but a toy model, in which some of the higher-dimensional properties can be explored. However, it seems absolutely necessary to incorporate these effects in more realistic scenarios emerging from string/M-theory. These scenarios will in general contain D-branes, which are much more complicated objects than orbifold branes. Not only are they not submitted to any reflection symmetry, but their physics is much richer due to the presence of gauge fields which have here been neglected. The extension of the study in this thesis to more realistic and richer scenarios is hence the real objective one should aim at.

This thesis is therefore only a small step in the study of cosmological signatures of braneworld cosmology. Due to the nature of the toy model we concentrated on, and because of the approximations we have made, the conclusion can only be limited. However, it has allowed us to develop different ways to attack the problem depending on which specific regime we are interested in. Understanding the problem as a whole is a very difficult task to address, but we have attempted to decompose it in small, better defined questions. For these specific questions, a set of tools has been developed which can then be used in more sophisticated situations. Applied to the RS toy models, not many new physical results can be expected, but we hope that the way of approaching the problem can be extended to more realistic scenarios where the physical implications would be richer.

CHAPTER 6. CONCLUSIONS

Appendix A

Covariant study of an empty bulk brane

In this appendix, we apply the same formalism as in section 3.2.1, in order to model the RS model in presence of a bulk brane as described in section 2.8.2. In particular, we start with the similar assumption that all branes may be treated as conformal to each other:

$$q_{\mu\nu}^{(\pm)} = X^{(\pm)\,2} q_{\mu\nu}^{(b)} \tag{A.1}$$

where we use the same notation as in section 2.8.2: the superscripts (\pm) designate the positive and negative boundary branes and quantities referring to the bulk brane are labelled with the index (b). Although this conformal assumption is not valid in general, we expect it to remain valid for long wavelength adiabatic perturbations, and in chapter 4, we explain how this conformal assumption may be broken for the usual RS scenario. The same procedure could be used to extend the bulk brane treatment that we shall explain in what follows.

In what follows we consider matter to be present only on the boundary branes, the bulk brane being empty. We will use the same notation as in section 3.2.1 and work in the bulk brane frame. Therefore all covariant derivatives and indices raising will be taken with respect to $q_{\mu\nu}^{(b)}$ unless specified otherwise. Considering the equation of motion (3.3) of the conformal factor in the two-branes model, the fields X and Y can be shown to satisfy

the analogous equation of motion:

$$\Box X^{(\pm)} = \frac{1}{6} \left(R^{(b)} - X^2 \Pi^{(\pm)} \right) X^{(\pm)}$$
 (A.2)

where $\Pi^{(\pm)}$ are defined in section 3.2.1. For empty boundary branes, we may point out that X and Y are conformally invariant with respect of the bulk brane.

The procedure used in section 3.2.1 in order to express the Weyl tensor covariantly as in (3.25) may be followed exactly the same way here. There are now two traceless and conserved pseudo-stress-energy tensor $T^{X,Y}_{\mu\nu}$ eff defined the same way as in (3.21 - 3.23):

$$T_{\mu\nu}^{X^{(\pm)} \text{ eff}} = G_{\mu\nu}^{(2)} X^{(\pm)\,2} + 4 \,\partial_{\mu} X^{(\pm)} \,\partial_{\nu} X^{(\pm)} - 2X^{(\pm)} D_{\mu} D_{\nu} X^{(\pm)}$$

$$+ q_{\mu\nu}^{(b)} \left(2X^{(\pm)} \Box X^{(\pm)} - \left(\partial X^{(\pm)} \right)^2 \right) - X^{(\pm)\,2} \tilde{\Pi}_{\mu\nu}^{(\pm)} + \mathcal{E}_{\mu\nu}^{(\pm)}$$

$$(A.3)$$

$$q^{(b)\mu\nu}T^{X^{(\pm)}}_{\mu\nu} \,^{\text{eff}} = 0 \tag{A.4}$$

$$D_{\mu}T^{X^{(\pm)}} \, \stackrel{\text{eff}\,\mu}{\nu} = 0 \tag{A.5}$$

where $\mathcal{E}_{\mu\nu}^{(\pm)}$ satisfy the equation (3.18), (3.19). For the background, following the procedure of section 2.8.2, the induced Weyl tensors on each brane give rise to the dark energy terms in the modified Friedmann equations (2.93):

$$E^{(+)i}_{\ \ j} = -\frac{1}{3} E^{(+)0}_{\ \ 0} \delta^i_j = -\frac{\mathcal{C}_L}{L_L^2} \frac{1}{a_+^4} \delta^i_j \tag{A.6}$$

$$E^{(-)i}_{\ \ j} = -\frac{1}{3} E^{(-)0}_{\ \ 0} \delta^{i}_{\ j} = -\frac{\mathcal{C}_{R}}{L^{2}_{R}} \frac{1}{a^{4}_{-}} \delta^{i}_{\ j}$$
(A.7)

$$\bar{E}^{(b)\,i}_{\ \ j} = -\frac{1}{3}\,\bar{E}^{(b)\,0}_{\ \ 0}\delta^{i}_{\ j} = -\frac{\mathcal{C}_{R}L_{L} - \mathcal{C}_{L}L_{R}}{L_{L}L_{R}\left(L_{R} - L_{L}\right)}\frac{1}{a_{b}^{4}}\,\delta^{i}_{\ j},\tag{A.8}$$

where we have considered the bulk brane to be empty. *In the background*, the induced Weyl tensor on each brane are therefore related as follows:

$$\bar{E}_{\mu\nu}^{(b)} = \frac{L_L}{L_L - L_R} X^{(+)\,2} E_{\mu\nu}^{(+)} + \frac{L_R}{L_R - L_L} X^{(-)\,2} E_{\mu\nu}^{(-)}.$$
(A.9)

By analogy to (3.5), we construct the traceless quantity $\Delta E_{\mu\nu}$ that vanishes in the background:

$$\Delta E_{\mu\nu} = \left(-\frac{L_L}{L_L - L_R} X^{(+)\,2} \Pi^{(+)}_{\mu\nu} - \frac{L_R}{L_R - L_L} X^{(-)\,2} \Pi^{(-)}_{\mu\nu} \right)$$

$$- \left(R^{(b)}_{\mu\nu} - \frac{L_L}{L_L - L_R} X^{(+)\,2} R^{(+)}_{\mu\nu} - \frac{L_R}{L_R - L_L} X^{(-)\,2} R^{(-)}_{\mu\nu} \right).$$
(A.10)

Using the same arguments as in section 3.2.1, the Weyl tensor projected onto the bulk brane shall be expressed in terms of the pseudo stress-energy tensors,

$$E_{\mu\nu}^{(b)} = -\frac{L_L}{L_L - L_R} T_{\mu\nu}^{X^{(+)} \text{ eff}} - \frac{L_R}{L_R - L_L} T_{\mu\nu}^{X^{(-)} \text{ eff}} + \mathcal{E}_{\mu\nu}^{(b)} + \Delta E_{\mu\nu}^{TT}$$
(A.11)

where the longitudinal part $\mathcal{E}_{\mu\nu}^{(b)}$ is introduced so that the Bianchi identity on the bulk brane (2.98) is satisfied:

$$\mathcal{E}_{\mu\nu}^{(b)} = D_{\mu}\mathcal{E}_{\nu}^{(b)} + D_{\nu}\mathcal{E}_{\mu}^{(b)} - \frac{1}{2}q_{\mu\nu}^{(b)}D_{\alpha}\mathcal{E}^{(b)\alpha}$$
(A.12)

$$D_{\mu} \mathcal{E}^{(b)\,\mu}_{\ \nu} = D^{(b)}_{\mu} \left(\bar{K}^{\mu}_{\alpha} \bar{K}^{\alpha}_{\nu} - \frac{1}{2} \bar{K}_{\alpha\beta} \bar{K}^{\alpha\beta} \delta^{\mu}_{\nu} \right). \tag{A.13}$$

The Weyl tensor is therefore covariantly known up to a traceless and conserved tensor $\Delta E_{\mu\nu}^{TT}$ which vanishes in the background. Following the same argument we used before, this tensor is negligible for the purpose of long wavelength adiabatic perturbations and shall be set to zero in what follows. In order to extend this treatment, we could include in $\Delta E_{\mu\nu}^{TT}$ the same kind of corrections as the one considered in chapter 4, but we will neglect this contribution for now.

Unlike in the case of \mathbb{Z}_2 symmetric branes, the Gauss-Codacci formalism can not be used to determine the extrinsic curvature entirely. The average part $\bar{K}_{\mu\nu}$ in (2.98) remains unknown despite the conformal assumption. However from its background behaviour, we might check that $\bar{K}_{\mu\nu}$ should be of second order in derivatives and from the Gauss-Codacci formalism this tensor is known to be conserved and traceless. Inspired by the covariant expression of the Weyl tensor in terms of the pseudo-stress-energy tensors $T^{X^{(\pm)}}_{\mu\nu}$ eff, these tensors represent a natural ansatz for the expression of $\bar{K}_{\mu\nu}$. A rough argument to justify this goes as follow: when the three branes are empty, up to second order in derivatives, the only local, conserved and traceless tensor that can be constructed from the theory (*ie.* from gravity with two scalar fields) without introducing new dimensionful parameters are the two stress-energy tensors $T^{X^{(\pm)}}_{\mu\nu}$ eff. When matter is included on the boundary branes, the tensors $T^{X^{(\pm)}}_{\mu\nu}$ eff as defined in (A.3) remain traceless and conserved. As $\bar{K}_{\mu\nu}$ satisfies the same properties, a natural ansatz is therefore a linear combination of these two tensors. Furthermore $\bar{K}_{\mu\nu}$ characterises the asymmetry around the bulk brane. It must therefore vanish when the reflection symmetry is locally imposed around the bulk brane: $q_{\mu\nu}^{(+)} = q_{\mu\nu}^{(-)}$ ie. when $X^{(+)} = Y^{(-)}$. Imposing $\bar{K}_{\mu\nu}$ to vanish when $X^{(+)} = Y^{(-)}$, forces the possible combination of these two tensors to be their difference. It is therefore reassuring to see that for the background, $\bar{K}_{\mu\nu}$ is precisely what we expect from this previous argument:

$$\bar{K}_{\mu\nu} = \gamma \left(T_{\mu\nu}^{X^{(+)} \text{ eff}} - T_{\mu\nu}^{X^{(-)} \text{ eff}} \right), \text{ with the constant } \gamma = \frac{1}{2} \frac{L_L L_R}{(L_L - L_R)}.$$
(A.14)

Covariantly we can therefore argue that the real expression for $\bar{K}_{\mu\nu}$ should be

$$\bar{K}_{\mu\nu} = \bar{K}^{(0)}_{\mu\nu} + \bar{K}^{TT}_{\mu\nu} \tag{A.15}$$

$$\bar{K}^{(0)}_{\mu\nu} = \frac{1}{2} \frac{L_L L_R}{(L_L - L_R)} \left(T^{X^{(+)} \text{ eff}}_{\mu\nu} - T^{X^{(-)} \text{ eff}}_{\mu\nu} \right)$$
(A.16)

where the tensor $\bar{K}_{\mu\nu}^{TT}$ includes higher that second order in derivatives, is conserved, traceless and vanishes for the background. For the study of adiabatic perturbations, it is therefore consistent to neglect its contribution. Beyond this approximation, $\bar{K}_{\mu\nu}^{TT}$ might include the same kind of corrections as expressed in sections 4.2.1 and 5.7. In particular, a natural ansatz for $\bar{K}_{\mu\nu}^{TT}$ would be the tensor $\alpha/L^2 A_{\mu\nu}$ (as defined in (4.6)), or the tensors $\beta/L^2 \hat{U} T_{\mu\nu}^{X^{(\pm)}}$ eff (where \hat{U} is defined in (5.173) and α and β are some dimensionless coefficients). We may note that in that case, the presence of these correction terms in the modified Einstein equation (2.98) will lead to terms of 6th and 8th order in derivatives.

Based on the Gauss-Codacci formalism, the conformal assumption for the Ricci tensor on the bulk brane can be found in terms of two scalar fields and the matter content of the boundary branes. For the purpose of long wavelength adiabatic linear perturbations, we argue that as long as the bulk brane is empty of matter and gauge fields, its metric should satisfy:

$$R_{\mu\nu}^{(b)} = \frac{1}{4} \frac{L_L^2 L_R^2}{(L_L - L_R)^2} \left(T^{X^{(+)} \text{ eff } \alpha}_{\mu} - T^{X^{(-)} \text{ eff } \alpha}_{\mu} \right) \left(T^{X^{(+)} \text{ eff } - T^{X^{(-)} \text{ eff }}_{\alpha\nu} - T^{X^{(-)} \text{ eff }}_{\alpha\nu} \right)$$
(A.17)
+ $\left(\frac{L_L}{L_L - L_R} T^{X^{(+)} \text{ eff }}_{\mu\nu} + \frac{L_R}{L_R - L_L} T^{X^{(-)} \text{ eff }}_{\mu\nu} - \mathcal{E}^{(b)}_{\mu\nu} \right)$
$$\Box^{(b)} X^{(\pm)} = \frac{1}{6} \left(R^{(b)} - X^{(\pm)\,2} \Pi^{(\pm)} \right) X^{(\pm)}$$
(A.18)

with $T_{\mu\nu}^{X^{(\pm)} \text{ eff}}$ as defined in (A.3).

The result of this procedure may again seem tedious to exploit. However for the purpose of linear perturbations we argue that the result will be reliable and will correctly take into account the characteristics features of its braneworld nature, namely the ρ^2 corrections on the boundary branes and the asymmetric terms present on the bulk brane.

It is straightforward to extend this model to a higher number of bulk branes. A less obvious generalisation of this prescription would be to add matter onto the bulk brane. In that case the average extrinsic curvature couples to the matter on the bulk brane in a highly non trivial way and its covariant expression seems less straightforward. We argue however that this procedure allows us to study the asymmetric terms present in a bulk brane in a original way and goes beyond the general treatment.

APPENDIX A. COVARIANT STUDY OF AN EMPTY BULK BRANE

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